EMBEDDING COMPACT RIEMANN SURFACES IN
4-DIMENSIONAL RIEMANNIAN MANIFOLDS

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Abstract. Any compact Riemann surface has a conformal model in any orientable Riemannian manifold of dimension 4. Precisely, we will prove that, given any compact Riemann surface, there is a conformally equivalent model in a prespecified orientable 4-dimensional Riemannian manifold. This result along with [10] now shows that a compact Riemann surface admits conformal models in any Riemannian manifold of dimension $\geq 3$.

1. Introduction

In 1989 and 2001, the author ([9, 10]) applied Teichmüller theory to prove that, for every compact Riemann surface $S_0$ and every orientable model manifold $M$ of dim $M \geq 3$, there is a model surface $S$ embedded in $M$ so that $S$ is conformally equivalent to the original Riemann surface $S_0$. See the historical discussion in [10] concerning the earlier work on conformal structure and conformal immersion. In [10], the case of a 4-dimensional $M$ required the extra technical assumption that the normal bundle have a nowhere vanishing cross-section. In the present paper we remove this assumption and thus conclude, with [10], that a compact Riemann surface now admits conformal embedding into any Riemannian manifold of dimension $\geq 3$.

In 1993, author ([11]) also extended this result to the finite topological type Riemann surface in Riemannian manifold $M$ of dim $M \geq 3$ with some restrictions on the embedding into 4-dimensional Riemannian manifold.

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In 1999, the author ([12]) extended the above theorems to any open Riemann surface in a prespecified orientable Riemannian manifold $\mathcal{M}$ of dim $\mathcal{M} \geq 3$ except the partial proof for the embedding into 4-dimensional Riemannian manifold.

We here give a complete proof that, given compact Riemann surface, there exists a conformally equivalent model in an orientable 4-dimensional Riemannian manifold.

In this paper, we refer lots of computations, definitions, lemmas and proofs to the previous work in [10] from time to time. So it is absolutely necessary to have a copy of [10] open while reading this paper.

2. The Main Results and the Guides to the Proof

2.1. The Main Results. We will see in this paper that the methods used in the above Ko Embedding Theorems are even strong enough to prove this theorem for embedding into 4-dimensional orientable Riemannian manifolds too.

Let $\mathcal{M}$ be an orientable Riemannian manifold of dim $\mathcal{M} = 4$ and $S$ be a closed $C^\infty$-embedded Riemann surface in $\mathcal{M}$.

We follow now carefully all the notations and arguments of [10] beginning with [10], section 2 to prove

**Embedding Theorem.** Assume that $S$ is a compact Riemann surface, $C^\infty$-embedded in the orientable Riemannian manifold $\mathcal{M}$ of dim $\mathcal{M} = 4$. Let $S_0$ be any Riemann surface structure on $S$. Then $S_0$ is conformally equivalent to a complete classical surface in $\mathcal{M}$. A model can be constructed by deforming a given topologically equivalent complete Riemann surface $S$ in the normal direction $NS$ of $S$.

We know that, in the case of codimension 2, there are sections of the normal bundle $NS$ of $S$ in $\mathcal{M}$ with isolated zeroes. (See Ko [9, p. 9] or [10, Section 2.2] for details on the sections of the normal bundle.) We consider this case only, otherwise the theorem is true([9, 10]).

**Remark.** It can be shown that if dim $\mathcal{M} \neq 4$, then there always exists a nowhere vanishing section of $NS$ if $S$ is compact. When dim $\mathcal{M} = 4$, the nowhere vanishing section of $NS$ exists if there are no obstructions. In this case the obstruction lies in the Euler class $e(NS)$ of the normal bundle $NS$. That is, if $e(NS) = 0$, then there is always such a section. For the proof see Ko [9].

The argument now continues as in [10], section 2. For the theory and the coordinate systems of Teichmüller space of a Riemann surface, we refer to [10],
section 3 to [10], section 4. We need several supporting lemmas, especially Garsia’s Continuity Lemma and (revised) Brouwer’s Fixed Point Lemma. We refer them to Lemma 4.1 (Garsia’s) and Lemma 5.2 (Brouwer’s) in [10], sections 4 and 5, respectively.

2.2. Guides to the Proof of the Embedding Theorem. We follow the same notations as in [10], section 2 to [10], section 4.

Let \( S \) be a compact Riemann surface \( \mathcal{C}^\infty \)-embedded in the orientable Riemannian manifold \( M \) of \( \dim M = 4 \) and \( S_0 \) be any Riemann surface structure on \( S \).

In the case of codimension 2, there are some (say \( n \)) fixed exceptional points where the section of \( N_S \) vanishes as indicated in the Section 2.1. But this problem can be resolved by constructing \( h \) be zero in the neighborhoods of those \( n \) exceptional points using a theorem of Bers.

By Nash’s embedding theorem, there is a \( \mathcal{C}^\infty \)-isometric embedding \( M \hookrightarrow \mathbb{R}^m \) for some sufficiently large \( m \). This allows us to consider \( S \) and \( M \) as subsets of \( \mathbb{R}^m \).

Assume that \( \tilde{S} \) is a holomorphic universal covering of \( S \). Let \( X : \tilde{S} \to \mathcal{M} \subset \mathbb{R}^m \) be a local conformal parametrization of \( S \) in \( \mathcal{M} \subset \mathbb{R}^m \). Then we may identify the point \( X(z) \in S \) with the point \( z \in \tilde{S} \).

Let \( \Gamma : S \hookrightarrow NS \setminus \Gamma_0 \) be a smooth section of \( NS \) which vanishes at some (say \( n \)) fixed exceptional points, where \( \Gamma_0 \) is the zero section of \( NS \). \( \Gamma \) has a maximum length 1.

Let \( \mathcal{T}(S) \) be a Teichmüller space of \( S \), then it can be identified with the open unit ball \( Q_1(S) \setminus \{0\} \). For any \( \omega = \phi_\omega(z)dz^2 \in Q_1(S) \setminus \{0\} \), define a metric \( ds^2_\omega \) on \( S \) by

\[
(2.1) \quad ds^2_\omega := \lambda^2(z) \left| dz + |\omega|^{1/2} \frac{\phi_\omega(z)}{|\phi_\omega(z)|} d\bar{z} \right|^2,
\]

where \( \lambda^2 > 0 \) is a smooth real-valued \((1,1)\)-form. This metric (2.1) defines a new conformal structure on \( S \), which will be denoted by \( S_\omega = (S, ds^2_\omega) \).

If \( f_\omega : S \to S_\omega \) is a quasiconformal map and \([f_\omega] \in \mathcal{T}(S)\), then write \([f_\omega] = \omega \).

Let \( f_0 : S \to S_0 \) be a homeomorphism such that \([f_0] \in \mathcal{T}(S)\). Assume that \([f_0] = \omega_0 \) and denote by \( B_\epsilon(\omega_0) \subset \mathcal{T}(S) \) the set of elements in \( \mathcal{T}(S) \) with \( \|\omega - \omega_0\| < \epsilon \).

Then Garsia’s Continuity Lemma ([10], Lemma 4.1) and (revised) Brouwer’s Fixed Point Lemma ([10], Lemma 5.2) follow.

In section 3 of this paper, we introduce a process which is defined when \( \omega \) is restricted to a compact subset of \( Q_1(S) \setminus \{0\} \). It uses a family of the metrics
\[ dX^2 = \lambda^2(z)|dz|^2 \] to generate a family of smooth deformations in \( \mathcal{M} \) of the surface \( S \).

First we fix a map \( h \) on \( S \times B_\epsilon(\omega_0) \) so that \( h \) is a \( C^\infty \) function and \( \| h \|_\infty < \epsilon \) on \( S \) for each fixed \( \omega \). We will construct this \( h \) be 0 in the neighborhoods of \( n \) fixed exceptional points so that the vanishing of sections of the normal bundle at finitely many exceptional points doesn’t effect the deformability. Denote by \( [S^\omega] = [[S^\omega, S^\omega]] \) the conformal equivalence class of the surface \( S^\omega \) as a marked surface \((S^\omega, S^\omega)\). We then define a map \( \Xi \) of \( B_\epsilon(\omega_0) \) to \( T(S) \) by

\[
\Xi : B_\epsilon(\omega_0) \rightarrow T(S) \quad \omega \mapsto [S^\omega].
\]

Here the surface \( S^\omega := S_h(\cdot, \omega) \) is the \( \epsilon \)-normal deformation of \( S \) in \( \mathcal{M} \) defined by the map

\[
(2.2) \quad S^\omega := S_h(\cdot, \omega) : S \rightarrow \mathcal{M} \subset \mathbb{R}^m \quad X(z) \mapsto X(z) + h(X(z), \omega)\tilde{\Gamma}(X(z)) + O(h^2),
\]

where \( \tilde{\Gamma}(X(z)) \) is a unit tangent vector in \( \mathbb{R}^m \) to the curve \( \exp th(X(z))\Gamma(X(z)) \) at the point \( X(z) \). (For more details, see Section 2 of \[10\].)

Then, as a consequence of (revised) Brouwer fixed point Lemma, we will have proved the existence of the conformal model if we can prove that, given \( [f_0] = \omega_0 \) and \( \epsilon > 0 \), for \( \omega \) in the closed ball \( B_\epsilon(\omega_0) \subset Q_1(S) \), there is a family of deformations \( S^\omega \) of \( S \) depending on parameters \( \omega \) so that the following is true.

**Lemma 2.1 (Dependence of \( S^\omega \) on Parameters \( \omega \)).** In the above notations,

1. \( \Xi \) : \( \omega \mapsto [S^\omega] \) is continuous in \( B_\epsilon(\omega_0) \).
2. \( \|[S^\omega] - [id_\omega]\| \leq \epsilon, \forall \omega \in B_\epsilon(\omega_0), \) where \( id_\omega : S \rightarrow S_\omega \) is the set-theoretic identity map.

**Proof.** Basically, the arguments are the same as those of Lemma 5.3, \[10\]. The proof requires the application of Garsia’s Continuity Lemma back and forth. But in this case, for the function \( \chi = S^\omega \circ (id_\omega)^{-1} : S_\omega \rightarrow S^\omega \), the computation of \( K_X \) needed to apply Garsia’s Continuity Lemma requires extra considerations other than that of Lemma 5.3 in Ko \[10\] due to the (different) expression of deformation function \( h \). We give the computation of \( K_X \) in Lemmas 3.1 and 3.2. \( \square \)

By Lemma 2.1, the function \( \Xi \) satisfies the hypotheses of (revised) Brouwer Fixed Point Lemma. Therefore there is a point \( \omega_1 \in B_\epsilon(\omega_0) \) so that

\[
\Xi(\omega_1) = [S^{\omega_1}] = \omega_0 = [f_0], \quad \text{where} \quad f_0 : S \rightarrow S_0,
\]
i.e., for this $\omega_1 \in B_\epsilon(\omega_0)$, the deformed surface $S_{\omega_1}$ can be mapped conformally onto $S_0$ by a mapping homotopic to $f_0 \circ (S^{\omega})^{-1}$.

Finally in Section 4, we collect all the facts needed to finish the proof of the Embedding Theorem.

3. Deformations of a compact Riemann surface

We will assume that the genus $g$ of a compact Riemann surface $S$ always is $\geq 1$ since genus 0 compact Riemann surfaces are all conformally equivalent. Let $G$ be a group of deck transformation on $S$ and $P$ be a fundamental domain for $G$. We may assume that $\partial P$ has measure zero and the fundamental domain $P$ is compact in $\tilde{S}$ since $S$ is compact (see Lehner [13, p. 203-205]). We will not have to worry about the parametrization being two-to-one along $\partial P$. We follow the notations of Section 2.2.

In several places we define two different functions, one in genus $g = 1$ and one in genus $g > 1$. We use the same notation for two functions with different meanings (for example, $\mu_\eta(x)$ for $g = 1$, $\mu_\eta(x, y, \omega)$ for $g > 1$) in order to obtain a treatment valid in all genera.

In view of the previous section, it is necessary to compare two metrics $(dS^\omega)^2$ and $ds^2_\omega$. From the equation (2.2), $(d\tilde{S}^\omega)^2 = (d\tilde{S}_h)^2$ satisfies the equation

$$(dS^\omega)^2 = (d\tilde{S}_h)^2 = \lambda^2(z)|dz|^2 + (dh)^2 + o(h)|dz|^2.$$

For $ds^2_\omega$ in (2.1), let

$$\Pi(\omega) = \{z | z \in \tilde{S}, \Im \phi_\omega(z) \neq 0\}$$

if $g > 1$ or $\Pi(\omega) = \mathbb{C}$ if $g = 1$. Then $ds^2_\omega$ is smooth on $\Pi(\omega)$. As in section 6.1 of [10], for appropriate (real-valued functions) $\gamma_\omega, \alpha_\omega$ and $\beta_\omega$, we may rewrite $ds^2_\omega$ on $\Pi(\omega)$ as

$$\gamma_\omega ds^2_\omega = \lambda^2(z) (|dz|^2 + (\alpha_\omega dx + \beta_\omega dy)^2).$$

3.1. The Deformation Function $h$ for $g = 1$. Following previous arguments, to complete the Embedding Theorem, we need to describe a deformation function $h$ on $S$ which satisfies the following properties:

1. $h$ is $C^\infty$.
2. $\|h\|_\infty < \epsilon$.
3. $(dh)^2$ is proportional to $(\alpha_\omega dx + \beta_\omega dy)^2$ in view of (3.1) and (3.2) (also see [10], section 6.1 for details).
We would like to define a function $h$ which satisfies condition 3 except on a sufficiently small set plus small neighborhoods of $n$ exceptional points (which are isolated zeroes of the smooth sections). We are tempted to write $h = \alpha_\omega x + \beta_\omega y$, but this function will usually violate condition 2. Therefore we must abandon global linearity and get a smooth approximation to a piecewise linear function $h$. We wish to apply Garsia’s Continuity Lemma([10], Lemma 4.1). The hypotheses of the Lemma would be simple if we could apply the saw-tooth function with slope $\pm 1$ to $\alpha_\omega x + \beta_\omega y$. The sign change doesn’t affect condition 3 except at the corners and this can be smoothed away on a small set. On the other hand, near the tip, condition 1 is violated. A smoothing procedure gives the necessary improvements. But then condition 3 is again broken. Again we smooth away the problem on a small set. For $h$ to be well-defined on $S$, it is convenient that it be zero in a neighborhood of the edges of $P$ and neighborhoods of $n$ exceptional points but remains smooth. (In this $g = 1$ case, without loss of generality, we take $P = [0,1]^2$.)

We now proceed to define the smoothing procedure. In Section 3.3, we will compute the necessary estimates. Suppose that $\eta$ is a fixed small number less than $\frac{1}{16}$.

We shall define two auxiliary real-valued differentiable functions as follows:

a. $\mu_\eta(x)$ :
   
   (1) $0 \leq \mu_\eta \leq 1$,
   
   (2) $\mu_\eta(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq \frac{\eta}{2} \\ 1 & \text{for } \eta \leq x \leq \frac{1}{2} \end{cases}$,
   
   (3) $\mu_\eta(1 - x) = \mu_\eta(x)$.

b. $\nu_\eta(x)$ :
   
   (1) $|\nu_\eta(x)| \leq 1$ (derivative $\nu_\eta(x)$ is taken with respect to the variables in it.)
   
   (2) $\nu_\eta(x) = \begin{cases} x & \text{for } \eta - 1 \leq x \leq 1 - \eta \\ 2 - x & \text{for } 1 + \eta \leq x \leq 3 - \eta \end{cases}$,
   
   (3) $\nu_\eta(x + 4) = \nu_\eta(x)$.

Next we define a function to take care of the $n$ fixed exceptional points where the section $\Gamma$ of the normal bundle $NS$ vanishes. Let, for $i = 1, \cdots, n$, $z_i \in P$ be fixed exceptional points and $E_i := E_\delta(z_i)$ be a small neighborhood of $z_i$ so that the area $E < k_E \cdot \eta$, where $E = \bigcup_{i=1}^{n} E_i$ and $k_E$ is a small constant ($< \frac{1}{4}$) depending on $E$. We define a $C^\infty$ function (Beltrami differential) $\nu(x,y)(\|\nu\|_\infty < 1)$ on $P$ so that its support lies in the complement of the set $E$. [This can be
done, using a theorem of Bers, by defining a $C^\infty$-function (Beltrami differential) $\nu_i(x, y)(||\nu_i||_\infty < 1)$ having a support on a complement of each $E_i$ and multiplying them all. (See [1] for more information.) Let $I_i$ be a small neighborhood containing $\overline{E}_i$ (closure of $E_i$), $i = 1, \cdots, n$, with the area $P \cap I < k_I \cdot \eta$, where $I = \cup_{i=1}^n I_i$ and $k_I$ is a small constant ($< \frac{1}{4}$) depending on $I \supset \overline{E}$. We define a function $\overline{U}(x, y)$ on $P$ using $\nu(x, y)$ as follow so that it is $C^\infty$ on $P$:

$$\overline{U}(x, y) = \begin{cases} 0 & (x, y) \in E \\ 1 & (x, y) \in P - I \quad (I \supset \overline{E}). \end{cases}$$

On $I - E$, we may define any $C^\infty$-function (since it does not really matter which form we use as you may see in Lemma 3.1 as long as its sup norm is less than 1 and it is expressed) in terms of $\nu(x, y)$ so that $U$ is $C^\infty$ and $||\overline{U}||_\infty \leq 1$ on whole $P$.

Let $F$ be a compact subset in $Q_1(S)$ which does not contain $0$.

Let $N_F$ be an integer to be determined later in this section and $\epsilon$ be the constant to be determined in the Section 4. (This $\epsilon$ guarantees the existence of $\epsilon$-normal deformation surface $S^\omega$. Refer to Theorem 2.1 of [10], section 2.)

Summing up all the functions and the conditions, for $(x, y) \in P$, $\omega \in F$ and $N > N_F + \frac{1}{\epsilon} \cdot \max_{z \in P} |\lambda(z)|$,

we define

$$h(x, y, \omega, N) = \frac{1}{N} \lambda(x, y)\overline{U}(x, y)\mu_\eta(x)\mu_\eta(y)\nu_\eta(N \cdot (\alpha_\omega x + \beta_\omega y)).$$

Then $h$ is $C^\infty$ on $P$ with $\|
abla h\|_\infty < \epsilon$ and continuous on $P \times F$. We get

$$(dh)^2 = \lambda^2\overline{U}^2(x, y)\mu_\eta^2(x)\mu_\eta^2(y)\nu_\eta^2(N \cdot (\alpha_\omega x + \beta_\omega y))(\alpha_\omega dx + \beta_\omega dy)^2 + o\left(\frac{1}{N}\right)|dz|^2,$$

where $o\left(\frac{1}{N}\right) \to 0$ uniformly on $P \times F$ as $N \to \infty$. This follows because all the previous functions are continuous on this compact set and $\frac{1}{N}$ multiplies them all.

In the Section 3.3, we will check the conditions of the Garsia’s Continuity lemma. So we need to show the area of exceptional set

$$A = \{(x, y) \in [0, 1]^2 | \overline{U}^2(x, y) \cdot \mu_\eta(x) \cdot \mu_\eta(y) \cdot \nu_\eta^2(N \cdot (\alpha_\omega x + \beta_\omega y)) \neq 1\}$$

can be made arbitrarily small.

Let

$$A_\mu = \{(x, y) \in [0, 1]^2 | \mu_\eta(x) \cdot \mu_\eta(y) \neq 1\}.$$

Then its Euclidean area satisfies

$$\text{area } A_\mu < 4\eta(1 - \eta)$$
whose proof is a trivial observation.

Also we know that
\[
\{(x, y) \in [0, 1]^2 | \mathcal{U}^2(x, y) \neq 1\} = P \cap I
\]
has area $< k_I \cdot \eta$ and the area of
\[
A_N = \{(x, y) \in [0, 1]^2 | \nu_\eta^2 (N \cdot (\alpha_\omega x + \beta_\omega y)) \neq 1 \}
\]
satisfies
\[
\text{area } A_N < 2\eta
\]
for $N > \frac{2\sqrt{2}}{\sqrt{||\omega||}}$ (see Ko [10], Lemma 6.2 for computations).

In the computation of area $A_N$, we restrict $\omega$ to a compact set $F$ in $Q_1(S)$ which does not contain 0. Therefore we can replace the condition $N > \frac{2\sqrt{2}}{\sqrt{||\omega||}}$ by $N > N_F > \frac{2\sqrt{2}}{\sqrt{||\omega||}}$. Then it is true, for $A = A_\mu \cup A_N \cup (P \cap I)$, that
\[
\text{area } A \leq (\text{area } A_\mu) + (\text{area } A_N) + (\text{area } P \cap I) < \eta(6 - 4\eta + k_I)
\]
for all $N > N_F$.

3.2. The Deformation Function $h$ for $g > 1$. At the beginning of Section 3.1, we listed the conditions that we want $h$ to satisfy. Condition 3 suggests that we express $(dh)^2$ in terms of constants $\alpha_\omega$ and $\beta_\omega$. In contrast to the case $g = 1$, in higher genera $\alpha_\omega$ and $\beta_\omega$ must be non-constant functions of $z$. The definition of $h$, on the other hand, will come as a solution of a differential equation in which $\alpha_\omega$, $\beta_\omega$ and their derivatives appear as coefficients. In order to get a $C^\infty$ solution, we need $\alpha_\omega$, $\beta_\omega$ to be smooth on all of $P$. Also they, together with their derivatives, must change as little as possible.

In this section, we will eventually construct the deformation function $h$ on $P$ for $g > 1$ in terms of $\lambda(z)$, $\alpha_\omega(z)$, $\beta_\omega(z)$ and some large number $N$.

For this purpose, we need to extend the functions $\alpha_\omega$ and $\beta_\omega$ on whole of $P$ since they are not defined on $P \setminus \Pi(\omega)$. This extension has been done in [10], section 6.4. There([10], sections 6.4.2 to 6.4.3) we set them as $\tilde{\alpha}_\omega$ and $\tilde{\beta}_\omega$, and constructed several other auxiliary functions on $\Delta \times F$ such as the real-valued continuous functions $\mu_\eta(z, \omega)$ (which is different from $\mu_\eta(x)$ in Section 3.1) and $u$, maximum value $u_0$ of $|u|$, an exact function $\varrho = e^{u_0}(\tilde{\alpha}_\omega dx + \tilde{\beta}_\omega dy)$ and differentiable function $k$ satisfying $\varrho = dk$.

We have defined a $C^\infty-$function $\tilde{\mathcal{U}}(x, y)$ with $||\tilde{\mathcal{U}}(x, y)||_\infty \leq 1$ on $P$ in (3.3) to take care of $n$ fixed exceptional points where the section $\Gamma$ of the normal bundle $NS$ of $S$ vanishes.
Let $N_F$ be an integer to be determined later in this section and let $\epsilon$ be the constant to be determined in the Section 4. (This $\epsilon$ guarantees the existence of $c$-normal deformation surface $S^\omega$. Refer to Theorem 2.1 of [10], section 2.) Summing up all the functions and the conditions, for $(x, y) \in P$, $\omega \in F$ and

$$ N > N_F + \frac{1}{\epsilon} \cdot \max_{z \in P} |\lambda(z)|, $$

we define

$$ h(x, y, \omega, N) = \frac{1}{N} \lambda(x, y) \mathcal{U}(x, y) \mu_\eta(x, y, \omega) e^{u(x, y, \omega) - u_0} \cdot \nu_\eta(N \cdot k(x, y, \omega)). $$

Then $h$ is a $C^\infty$ function on $P$ with $\|h\|_\infty < \epsilon$ and continuous in $\omega \in F$. For $\omega$ fixed, we obtain

$$ dh^2 = \lambda^2 \cdot \mathcal{U}^2 \cdot \mu_\eta^2 \cdot \nu_\eta^2 \cdot (\tilde{\alpha}_\omega dx + \tilde{\beta}_\omega dy)^2 + o\left(\frac{1}{N}\right)[dz]^2. $$

In the Section 3.3, we will check the conditions of the Garsia’s Continuity Lemma. So we still want to show the area of

$$ A = \{(x, y) \in P \mid \mathcal{U}^2(x, y) \cdot \mu_\eta(x, y, \omega) \cdot \nu_\eta^2(N \cdot k(x, y, \omega)) \neq 1\} $$

can be made arbitrarily small. If we let

$$ B = \{(x, y) \in P \mid \mu_\eta(x, y) \neq 1\} $$

then $B$ has an area([10], section 6.4.2)

$$ \text{area } B < \frac{\eta}{2}. $$

Let

$$ A_1 := \{(x, y) \in P \mid \nu_\eta^2(N \cdot k(x, y, \omega)) \neq 1\} \quad \text{and} \quad A_2 := \{(x, y) \in P \mid \mathcal{U}^2(x, y) \neq 1\}, $$

so that $A = B \cup A_1 \cup A_2$. Since we know that

$$ \text{area } A_2 = \text{area } P \cap I < k_I \cdot \eta, $$

we only need to determine the area of the set $A_1$. But we have computed the area of $A_1$ in (6.46) of [10], section 6.4.3 as

$$ \text{area } (A_1) < k_F \cdot \eta. $$

Summing up all these values, we obtain

$$ \text{area } A = \text{area } (B \cup A_1 \cup A_2) < \left(\frac{1}{2} + k_F + k_I\right) \eta. $$

Here we take $N_F > 4(k_F \sigma - 4k_0)$ so that the inequality (3.11) is valid for $N > N_F + \frac{1}{\epsilon} \cdot \max_{z \in P} |\lambda(z)|$, where $k_0$ is a maximum value of $|k|$. 
3.3. Continuity of the Metric \((dS^2)^2\) and the Computation of \(K_\chi\).  

We will need the following lemmas to complete the proof of Lemma 2.1 and the Embedding Theorem. Here we use the facts and computations in section 6.3, 6.4.2 and 6.4.3 of Ko [10] with slight modifications.

Lemma 3.1. Given \(h = h(x, y, \omega, N)\) as in equations (3.4) and (3.9), \(ds_\omega^2\) in (2.1) and \(\gamma_\omega ds_\omega^2\) in (3.2), we get

\[
\left| dh^2 + \lambda^2 |dz|^2 - \gamma_\omega ds_\omega^2 \right| \leq \begin{cases} 
R(\eta; N)ds_\omega^2 & \text{on } P - A, \ \omega \in F \\
\tilde{R}(\eta; N)ds_\omega^2 & \text{on } A, \ \omega \in F,
\end{cases}
\]

where the area of \(A\) is given by (3.8) (if \(g = 1\)) or (3.11) (if \(g > 1\)). The inequalities are valid for \(N > N_F + \frac{1}{4} \cdot \max_{z \in F} |\lambda(z)|\) where \(N_F\) is a constant depending on the compact set \(F\). For each fixed \(\eta, R(\eta; N)\) can be made arbitrarily small as \(N \to \infty\) and \(\tilde{R}(\eta; N)\) is some constant which is bounded as a function of \(N\).

Proof. (1) In genus \(g = 1\), we get

\[
| dh^2 + \lambda^2 |dz|^2 - \gamma_\omega ds_\omega^2 | \leq \begin{cases} 
R(\eta; N)ds_\omega^2 & \text{on } P - A, \ \omega \in F \\
\tilde{R}(\eta; N)ds_\omega^2 & \text{on } A, \ \omega \in F,
\end{cases}
\]

On \(P - A\), we have \(\mu_\eta(x) = \mu_\eta(y) = \tilde{\nu}_\eta(x) = 1\) and \(\tilde{\Omega}^2(x, y) = 1\) (equations (3.6) and (3.7)), so the right hand side of equation (3.12) becomes

\[
\left| o\left(\frac{1}{N}\right) |dz|^2 \right| \leq R(\eta; N)ds_\omega^2
\]

for some small constant \(R(\eta; N)\).

On \(A\), we have \(\mu_\eta^2(x) \cdot \mu_\eta^2(y) \neq 1\) and \(\tilde{\nu}_\eta^2 \neq 1\) or \(\tilde{\Omega}^2(x, y) \neq 1\) (equations (3.6) and (3.7)), so the right hand side (RHS) of equation (3.12) becomes

\[
\text{RHS} \leq \gamma_\omega \left( \tilde{\Omega}^2(x, y) \mu_\eta^2(x) \mu_\eta^2(y) \tilde{\nu}_\eta^2 - 1 + o\left(\frac{1}{N}\right) \right) ds_\omega^2
\]

(3.13)

for some constant \(\tilde{R}(\eta; N)\) which is not necessarily very small.

(2) In genus \(g > 1\), use \(\tilde{\alpha}_\omega\) and \(\tilde{\beta}_\omega\) instead of \(\alpha_\omega\) and \(\beta_\omega\), we obtain

\[
\left| dh^2 + \lambda^2 |dz|^2 - \gamma_\omega ds_\omega^2 \right| \leq \begin{cases} 
R(\eta; N)ds_\omega^2 & \text{on } P - A, \ \omega \in F \\
\tilde{R}(\eta; N)ds_\omega^2 & \text{on } A, \ \omega \in F,
\end{cases}
\]

(3.14)
On $P - A$, we have $\mu^2_n = \nu^2_n = 1$ and $\overline{\Omega}^2(x, y) = 1$ (equation (3.10) and thereafter), so the right hand side of equation (3.14) becomes

$$\left| o\left(\frac{1}{N}\right)dz \right|^2 \leq R(\eta; N)ds^2_\omega$$

for some small constant $R(\eta; N)$.

On $A$, since $\mu^2_n \cdot \nu^2_n \neq 1$ or $\overline{\Omega}^2(x, y) \neq 1$, the right hand side (RHS) of equation (3.14) becomes

$$\text{(3.15)} \quad \text{RHS}' \leq \left| \gamma_\omega \left( (\overline{\Omega}^2 \cdot \mu^2_n \cdot \nu^2_n - 1) + o\left(\frac{1}{N}\right) \right) ds^2_\omega \right| \leq \tilde{R}(\eta; N)ds^2_\omega$$

for some constant $\tilde{R}(\eta; N)$ which is not necessarily very small. □

For the $\epsilon -$ normal deformation $S^\omega$ of $S$, we have the following estimates.

**Lemma 3.2.** Given $h(x, y, \omega, N)$ by either (3.4) or (3.9) where the sup and the inf are taken over all directions at a point $z$. Then, for the deformed surface $S^\omega := S_h(x, \omega)$ (defined by (2.2)), we get, for $\omega \in F$ and each fixed $\eta$,

1. $$\left| (dS^\omega)^2 - dh^2 - dX^2 \right| \leq c(\eta; N)ds^2_\omega, \text{ where } c(\eta; N) \to 0 \text{ as } N \to \infty.$$
2. $$\left( \sup \frac{(dS^\omega_m)^2}{(dS^\omega)^2} \right) / \left( \inf \frac{(dS^\omega_m)^2}{(dS^\omega)^2} \right) \to 1 \text{ as } \omega_m \to \omega.$$
3. $$K^2_\chi = \left( \sup \frac{(dS^\omega)^2}{ds^2_\omega} \right) / \left( \inf \frac{(dS^\omega)^2}{ds^2_\omega} \right) \leq \begin{cases} 1 + c_1(\eta; N) & \text{on } P - A \\ 4\gamma_\omega + c_2(\eta; N) & \text{on } A, \end{cases}$$

where the constant $c_1$ can be made arbitrarily small for each fixed $\eta$ and sufficiently large $N$. $c_2$ is some constant which is not necessarily small. The area of $A$ is given by (3.8) (if $g = 1$) or (3.11) (if $g > 1$).

**Proof.** Ko [10], Lemmas 6.9 and 6.10. □

4. **Proof of the Embedding Theorem**

So far we have checked every condition we need in the hypotheses of Gar- sia’s Continuity Lemma for some compact set $F$ in $Q_1(S)$. Therefore if we take $\epsilon = \frac{1}{2} \min \{1 - \|\omega_0\|, \|\omega_0\|\}$ and $F = \overline{B}_\epsilon(\omega_0) \subset Q_1(S) \setminus \{0\}$, then we may now complete the proof of the Embedding Theorem by the arguments given in Section 2.2.

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