In this short paper, we introduce a geometry of discrete quasiconformal groups. This subject has been studied by several mathematicians, name them few, P. Tukia, G. Martin, F. Gehring, D. Sullivan. This is an application of the quasiconformal mappings.

For $n \geq 2$, we let $\mathbb{R}^n$ denote euclidean $n$–space, $\mathbb{R}^n$ its one point compactification $\mathbb{R} \cup \{\infty\}$ and $e_1, \cdots, e_n$ the standard orthonormal basis for $\mathbb{R}^n$.

A Möbius transformation acting on $\mathbb{R}^n$ is a finite composition of reflections in spheres and hyperplanes ; we let $\text{Mob}(n)$ denote the group of all such transformations. Denote $M(n)$ a group of orientation preserving Möbius transformations which is a subset of $\text{Mob}(n)$.

Stereographic projection $p$ is the mapping from $\mathbb{R}^n$ into the unit sphere $S^n$ in $\mathbb{R}^{n+1}$ given by
\[ p(x) = e_{n+1} + \frac{2(x - e_{n+1})|x - e_{n+1}|^2}{|x - e_{n+1}|^2}. \]

We define the chordal distance between two points $x$ and $y$ in $\mathbb{R}^n$ as
\[ q(x, y) = |p(x) - p(y)|, \]
and let $B_q(x, r)$ denote the chordal ball
\[ B_q(x, r) = \{ y \in \mathbb{R}^n ; q(x, y) < 1 \}. \]

Throughout this paper, the topology of $\mathbb{R}^n$ and all notions of convergence will be taken with respect to the chordal metric.

**Definition 1.** Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism. We say that $f$ is quasiconformal if distortion bounds spheres by a bounded amount (infinitesimally).

\[ 1 \leq H_f(x) = \lim_{r \to 0} \frac{\max_{|h|=r} |f(x + h) - f(x)|}{\min_{|h|=r} |f(x + h) - f(x)|} K. \]

$K$ is called the dilatation of $f$.

Then we get the following properties;

1. If $n = 2$, then $f$ is a linear fractional transformation.
2. If $f$ is quasiconformal, then $f$ is differentiable a.e., and the derivatives are in $L^{n+1}_{\text{loc}}$.
3. If $f$ is quasiconformal, then $f^{-1}$ is quasiconformal.
4. A composition of two quasiconformal mappings is quasiconformal.
Theorem 1 (Weyl and Gehring) Suppose that $D, D'$ are domains in $\mathbb{R}^n$ and that $f : D \to D'$ is a homeomorphism. If $n = 2$, then $f$ is 1-quasiconformal if and only if $f$ or its complex conjugate is a meromorphic function of a complex variable in $D$. If $n \geq 3$, then $f$ is 1-quasiconformal if and only if $f$ is the restriction to $D$ of a M"obius transformation, i.e., the composition of a finite number of reflections in $(n-1)$-spheres and planes.

(There are very few quasiconformal maps in higher dimensions.)

M"obius or Kleinian groups

Denote $\mathbb{R}^n = \partial H^{n+1}$, $H^{n+1} = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}, \ x_{n+1} > 0\}$, $ds^2 = |d\mathbf{z}|^2 x_{n+1}^{-1}$.

Definition 2 Suppose that $G$ is a group of self homeomorphisms of a domain $D$ in $\mathbb{R}^n$. We say that $G$ is discrete if it contains no infinite sequence of distinct elements which converge uniformly on compact subsets in $D$ to an element of $G$.

The group $G$ is said to be discontinuous at a point $x \in D$ if there is a neighborhood $U$ of $x$ in $D$ such that

$$g(U) \cap U = \emptyset$$

for all but finitely many $g \in G$. We denote $O(G)$ the set of all $x \in D$ at which $G$ is discontinuous and call $L(G) = \overline{D(G)} \setminus O(G)$ the limit set of $G$; we say that $G$ is discontinuous if $O(G)$ is not empty. It is easy to check that a discontinuous group is discrete; the converse is not true. (For example see Beardon [2] p. 96.)

We say that $G$ is properly discontinuous if the point $x \in D$ has a neighborhood $U$ such that $g(U) \cap U \neq \emptyset$ implies $g = id$. When $n = 2$, if $O(G) \neq \emptyset$, then we call $G$ a Kleinian group and Kleinian group $G$ is called Fuchsian if all its loxodromic elements are hyperbolic and $G$ leaves a disk or a half plane invariant.

Classical Theorem 2 Every orientable surface $S \neq T^2, C, C \setminus \{0\}, \hat{C},$ is homeomorphic to $H^2/T$, $\Gamma$ is a Kleinian group.

Note here that every $n$-dimensional Riemannian manifold $M^n$ of constant curvature is of this form $M^n \simeq H^n/\Gamma$, $\Gamma \subset \text{M"ob}(n)$, and $H^n$ is $\Gamma$-invariant.

M"obius transformations are hyperbolic isometries as matrix groups, i.e., groups of isometries of arbitrary metrics. For each $n$, we have

1. $n = 2$: $PSL(2, C)$
2. $n = 1$: $PSL(2, R)$.

In $H^{n+1}$, they can be generalized to Lie group theory.

In general, $\text{M"ob}(n+1) \simeq SO(n, 1)$ invariant under $ds^2 = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$.

We think of Kleinian groups as conformal groups and extend them to following quasiconformal groups.

Definition 3 Suppose that $G$ is a group of self-homeomorphisms of a domain $D$ in $\mathbb{R}^n$. Then $G$ is a $K$–quasiconformal group if each $g \in G$ is $K$–quasiconformal in $D$, $G$ is a quasiconformal group if it is $K$–quasiconformal for some $K$. By the general form of Liouville theorem (Theorem 1), a 1-quasiconformal group is in fact
the restriction to $D$ of a Möbius group when $n \geq 3$. That is a quasiconformal group is a group which acts with bounded distortion ($\log K$). (Conformal groups have no dilatation.)

**How might such quasiconformal groups arise?**

**Question** Are all quasiconformal groups the quasiconformal conjugate of some conformal (Möbius) groups $M(n)$ (which are subgroups of $M\text{öb}(n)$ consisting of all orientation preserving Möbius transformations in $M\text{öb}(n)$)? That is, if $G$ is a quasiconformal group of $\mathbb{R}^n$, then is $G$ of the form

$$G = fHf^{-1}$$

for some group $H$ of Möbius transformations of $\mathbb{R}^n$ and for some quasiconformal homeomorphism $f$ of $\mathbb{R}^n$? Clearly, all groups of this form are quasiconformal groups. ($\ast$)

**Theorem 3 (Sullivan)**

Every closed $n$–manifold admits a quasiconformal structure (coordinate changes are quasiconformal). If $n \neq 4$, this structure is unique. If $n = 4$, uniqueness is false although the existence is still true. (There exist different structure in the same manifold - Sullivan and Donaldson.)

**Examples**

Let $M^n$ be a closed $n$–manifold and $\tilde{M}^n_{q.c.}$ be a universal cover which is quasiconformal.

If

$$\tilde{M}^n_{q.c.} \overset{i(q.c.)}{\longrightarrow} \mathbb{R}^n$$

and if $\partial(i(M^n))$ is regular (that is admits a tangent plane) at a single point, then

$$\tilde{M}^n_{q.c.} \overset{q.c.}{\cong} B^n$$

and $M^n$ has the homotopy type of a hyperbolic space form. (For example, the assumption implies $((\text{curv } M)(x) < 0)$.

**Measurable Riemann Mapping Theorem**

If $f : D \to D'$ is quasiconformal, then $f$ has a nonsingular differential $df : \mathbb{R}^n \to \mathbb{R}^n$ at almost all $x \in D$. At each $x$, $df = df(x)$ maps an ellipsoid $E_f = E_f(x)$ about 0 with minimum axis length 1 onto an $(n-1)$–sphere about 0. Then $H_f(x)$ is the maximum axis length of $E_f(x)$ and the maximum stretching under $f$ at $x$ occurs in the directions of the smallest axes of $E_f(x)$. If $g : D' \to D''$ is quasiconformal, then $g$ is conformal if and only if $E_{gof} = E_f$ a.e. in $D$, and $E_f$ determines $f$ up to postcomposition with a conformal mapping.

When $n = 2$ and $f$ is sense preserving, then $E_f$ is determined by the Beltrami coefficient or complex dilatation

$$\mu_f(x) = f_x/f_x, \quad x = x_1 + ix_2$$
of \( f \) at \( x \). In particular, \( \mu_f \) is measurable with
\[
|\mu_f(x)| = \frac{H_f(x) - 1}{K(f) + 1} < 1,
\]
and \( \mu_{g \circ f} = \mu_f \) a.e. in \( D \) if and only if \( g : D' \to D'' \) is conformal. Moreover, in dimension two it is possible to prescribe the dilatation \( \mu_f \), and hence the ellipse \( E_f \), at almost every \( x \in D \).

**Theorem 5** (Measurable Riemann Mapping Theorem) If \( \mu \) is measurable with \( \|\mu\|_{L^\infty} < 1 \) in \( \mathbb{R}^2 \), then there exists a quasiconformal self mapping \( f = f_\mu \) of \( \mathbb{R}^2 \) with \( \mu_f = \mu \) a.e. If \( f \) is normalized to fix three points, then \( f \) is unique and depends holomorphically on \( \mu \).

Back to the question \((\ast)\)

1. For \( n = 1 \), quasiconformal group on the unit circle \( S^1 \) (essentially distorts cross ratio by a bounded amount.) ; unknown but true (True in simple cases ; abelian, not discrete)
   - Affirmative solution to \((\ast)\) : Nielsen realization problem (Kerckhoff) ; given \( \Gamma \) acting on \( S^1 \), \( \Gamma_0 = M\ddot{o}b \) of finite index in \( \Gamma \) is conjugate to Fuchsian group.

2. For \( n = 2 \), Yes by Sullivan - Tukia (1978, Stony Brook) ([8], [12])
   - Any Möbius transformation is conjugate to Fuchsian group, i.e., is of the form \( f^\Gamma f^{-1} \), where \( f \) is a quasiconformal map and \( \Gamma \) is a Fuchsian group.

**Theorem 6** When \( n = 2 \), each quasiconformal group \( G \) can be written in the form \( G = f^{-1} H f \), where \( H \) is a Möbius group and \( f \) a quasiconformal self-mapping of \( \mathbb{R}^2 \).

The Idea of Proof is to construct, for a given quasiconformal group \( G \), a \( G \)-invariant measurable Riemann structure in which \( G \) acts conformally, that is a measurable map \( \mu : \mathbb{R}^2 \to S \), where \( S \) is the space of positive definite symmetric \( n \times n \) matrices with determinant 1, such that for each \( g \in G \),
\[
\mu(x) = |\det g'(x)|^{-2/n} \cdot g'(x)^t \mu(g(x))g'(x).
\]

In this case \((n = 2)\), the measurable Riemann mapping theorem (Theorem 5) implies that this structure is in fact the pull-back, under a quasiconformal mapping, of the standard conformal structure, and so the group \( G \) can be conjugated by a quasiconformal mapping, so as to be conformal.

3. When \( n \geq 3 \) : Yes and No.
   - (No) : Tukia constructed ([9]) a quasiconformal group which is not isomorphic as a topological group and hence not quasiconformally conjugate, to a Möbius group. (Connected solvable Lie group of quasiconformal maps of \( \mathbb{R}^n \), \( n \geq 3 \), which is not discrete.) Later Martin ([6]) shows that discrete groups which are not quasiconformally conjugate to a conformal group.
   - For \( K > 1 \), every \( K \)-quasiconformal group are not quasiconformally conjugate to a Möbius group (McKemie).
(Yes) No measurable Riemann mapping theorem is true in higher dimensions, however Tukia ([10]) has shown that if the measurable $G$–invariant Riemann structure is continuous at a limit point of the group (or approximately continuous at a conical limit point), then the group is in fact the quasiconformal conjugate of a Möbius transformation.

Nevertheless, the following convergence property allows one to establish quasiconformal analogues of many properties of Möbius groups.

**Theorem 7** If $G$ is a discrete quasiconformal group, then for each sequence of distinct elements in $G$ there exists a subsequence $\{g_j\}$ and points $x_0, y_0$ in $\mathbb{R}^n$ such that $g_j \to y_0$ locally uniformly in $\mathbb{R}^n \setminus \{x_0\}$ and $g_j^{-1} \to x_0$ locally uniformly in $\mathbb{R}^n \setminus \{y_0\}$.

Suppose that $G$ is a group of self homeomorphism of $\mathbb{R}^n$. We say that $G$ is a *discrete convergence group* if it satisfies the conclusion of Theorem 7, and that an element $g$ of $G$ is *elliptic* if it is of finite order or periodic, and *parabolic* or *loxodromic* if it has infinite order and one or two fixed points, respectively.

**Theorem 8** Suppose that $G$ is a discrete convergence group. Then each element of $G$ is elliptic, parabolic, or loxodromic, and the limit set $L(G)$ is nowhere dense or equal to $\mathbb{R}^n$. Moreover if $\text{card} \left( L(G) \right) > 2$, then $L(G)$ is perfect, $L(G)$ lies in the closure of each nonempty $G$–invariant set, and the set of fixed point pairs of loxodromic elements in $G$ is dense in $L(G) \times L(G)$.

Though discrete convergence groups resemble Möbius groups in many respects, examples exists which show that they need not be topologically conjugate to möbius groups [4]. They also occur quite naturally in situations which have nothing to do with Möbius or quasiconformal groups.

**Theorem 9** A group $G$ of self homeomorphisms of $\mathbb{R}^n$ is a discrete convergence group if it is properly discontinuous in $\mathbb{R}^n \setminus E$, where $E$ is closed and totally disconnected.

It will be interesting to see how much of the classical theory of Kleinian groups carries for this class of groups.

Some of the results mentioned above have important applications in differential geometry. For example, Theorem 1 and the theorem on boundary correspondence of quasiconformal maps are key steps in the original proof of Mostow’s rigidity theorem [11].

**Theorem 10** If $n > 2$ and if $M$ and $M'$ are diffeomorphic compact Riemannian $n$–manifolds with constant negative curvature, then $M$ and $M'$ are conformally equivalent.

Later this theorem has been extended in several ways by several mathematicians ([8], [11]).
REFERENCES


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