EMBEDDABILITY OF RIEMANN SURFACES IN RIEMANNIAN MANIFOLDS

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ABSTRACT. In the earlier study of Riemann surfaces, one was interested in the embeddability of Riemann surfaces in Euclidean spaces. Complex Analysts showed that any Riemann surface can be embedded in Euclidean 3-space. But recently, this problem has been conveyed to the problem in the Riemannian manifolds. In this paper we investigate the history and very recent results of problem in affordable and systematic manner.

0. Introduction

$\mathcal{C}^\infty$-embedded surfaces are called classical surfaces if they are viewed as Riemann surfaces whose conformal structure is given in the following natural way: the local coordinates are those which preserve angles and orientation. The existence of the local coordinates is highly non-trivial, it means solving the Beltrami differential equation. This was done for analytic embeddings by Gauss, for differential embeddings by Korn-Lichtenstein. Because of the fundamental importance of this problem in the theory of quasiconformal mappings, it was investigated more thoroughly in 1950’s. For the most elegant treatment see [3].

In his lectures, Felix Klein emphasized that classical surfaces should be viewed as Riemann surfaces, i.e., as domains of analytic functions and integrals. In 1882, Klein posed the question of whether every Riemann surface is conformally equivalent to a classical surface (see Klein [6, p. 635]).

The earliest results in this direction were that every compact surface of genus zero is conformally equivalent to the sphere, every non-compact planar classical surface is equivalent to a subregion of the plane, and a compact classical surface of genus 1 is conformally equivalent to a torus of revolution provided its modulus is purely imaginary ([7]).

The first result beyond these facts was obtained by Teichmüller ([19]). He deformed an embedded surface by moving each point in the normal direction and studied the dependence of the conformal structures of the perturbed surface on deformation parameters.

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Around 1960, A. Garsia ([5]) proved that every compact Riemann surface can be conformally immersed in Euclidean 3-space $\mathbb{R}^3$. He failed to prove the embeddability by doing the deformations consecutively at the end point of each deformation. But he stated that he had found a realization of every compact Riemann surface as a classical surface although Klein required that classical surfaces be embedded. Garsia’s proof uses Teichmüller’s idea, results, and constructions inspired by Nash’s embedding theorem and Brouwer’s fixed point theorem.

In 1970, Rüdey extended Garsia’s result to open Riemann surfaces $S$ by applying Garsia’s techniques to compact exhaustions of $S$ ([16]). He also proved that every compact Riemann surface can be conformally embedded in $\mathbb{R}^3$ ([17, 18]).

In recent developments in mathematics and particle physics, the problem of embeddability of Riemann surfaces in Riemannian Manifolds has been aroused. Embedded Riemann surfaces occur in string theory—the so called theory of everything—as the world sheets, that is the trajectories, of strings moving in space-time. The strings are permitted to join and separate. In general, these surfaces are non-compact and have positive genus.

In 1989, the author ([7, 8]) used Teichmüller theory to prove that, for every compact Riemann surface $S_0$ and every orientable model manifold $M$ of dim $M \geq 3$, there is a model surface $S$ embedded in $M$ so that $S$ is conformally equivalent to the original Riemann surface $S_0$.

In 1993, the author ([11]) also extended this result to the finite topological type Riemann surface in Riemannian manifolds.

Most recently author ([12]) has shown the conformal embeddability of open Riemann surfaces in Riemannian Manifolds.

Present paper is an expository paper combining all the previous and very recent results clearly and systematically for easy availability of all in one with consistency. But we state the main streams and leave details to references in several places.

1. The Main Results

Let $M$ be an orientable Riemannian manifold of dim $M \geq 3$ and let $S$ be a closed $C^\infty$-embedded Riemann surface in $M$. We briefly examine one method of constructing deformations of $S$ in $M$.

Let $\Gamma : S \hookrightarrow NS \setminus \Gamma_0$ be a nowhere vanishing smooth section (with unit length) of the normal bundle $NS$ of $S$ in $M$, where $\Gamma_0$ is the zero section of $NS$. Let $h : S \rightarrow (-\epsilon, \epsilon)$ be a $C^\infty$-function on $S$ and call $\{h(x)\Gamma(x)\}$ a normal vector field on $S$. 
Let $\mathcal{M}_1$ be the subset of $\mathbb{N} \times \mathbb{S}$ consisting of all pairs

$$(x, r) := (x, r\Gamma(x)) \text{ for all } x \in \mathbb{S}, \text{ where } |r| < 2\epsilon.$$ 

Then $\mathcal{M}_1$ contains the pair $(x, h(x)\Gamma(x))$. Also let $\mathcal{M}_2$ be the set of all points

$$\{y \in \mathcal{M} : y = \exp r\Gamma(x), r \in (-2\epsilon, 2\epsilon), (x, r) \in \mathcal{M}_1\},$$

then $\mathcal{M}_2$ is a Riemannian submanifold of $\mathcal{M}$ for $\epsilon$ sufficiently small. Again, for sufficiently small $\epsilon$, the map $\beta : \mathcal{M}_1 \to \mathcal{M}_2$, defined by the exponential map $\beta(x, r) = \exp r\Gamma(x)$, is a diffeomorphism. This diffeomorphism depends on either $\mathbb{S}$ being compact or on $\mathcal{M}$ having some uniform bounds on its local geometry.

By Nash’s Embedding Theorem, there is a $\mathcal{C}^\infty$–isometric embedding $j : \mathcal{M} \to \mathbb{R}^m$ for some sufficiently large $m$. This allows us to consider $\mathbb{S}$ and $\mathcal{M}$ as subsets of $\mathbb{R}^m$.

Assume that $\bar{\mathbb{S}}$ is the holomorphic universal covering of $\mathbb{S}$. Let $X : \bar{\mathbb{S}} \to \mathcal{M} \subset \mathbb{R}^m$ be a local parametrization of $\mathbb{S}$ in the orientable Riemannian manifold $\mathcal{M} \subset \mathbb{R}^m$.

For $X(z) \in \mathcal{S}$, let

$$\alpha_{X(z)} : (-2, 2) \to \mathcal{M} \subset \mathbb{R}^m$$

$$t \mapsto \beta(X(z), t\hbar(X(z))) := \exp t\hbar(X(z))\Gamma(X(z)).$$

Then $\alpha_{X(z)}(t)$ is a $\mathcal{C}^\infty$–curve for which $\alpha_{X(z)}(0) = X(z)$ and

$$\alpha_{X(z)}(1) = \beta(X(z), h(X(z))) = \exp h(X(z))\Gamma(X(z)).$$

Let $\tilde{\Gamma}(X(z)) \in T_{X(z)}\mathbb{R}^m$ be a unit tangent vector in $\mathbb{R}^m$ to the curve $\alpha_{X(z)}(t)$ at the point $X(z) \in \mathbb{S}$. We then have

$$\alpha_{X(z)}(1) = X(z) + h(X(z))\tilde{\Gamma}(X(z)) + O(h^2(X(z))), \text{ } |h(X(z))| < \epsilon.$$ 

This is precisely the statement that $h(X(z))\tilde{\Gamma}(X(z)) + O(h^2(X(z)))$ is the tangent vector in $\mathbb{R}^m$ to the curve $\alpha_{X(z)}(t)$ at the point $\alpha_{X(z)}(0) = X(z)$.

Now, for any given sufficiently small $\epsilon > 0$, we define a normal deformation of $\mathbb{S}$.

**Definition 1.** In the above notation, a surface $\mathbb{S}_h \hookrightarrow \mathcal{M}$ is called an $\epsilon$–normal deformation of $\mathbb{S} \hookrightarrow \mathcal{M}$ if, for a given small $\epsilon > 0$, $h$ is a $\mathcal{C}^\infty$–real-valued function on $\mathbb{S}$ such that $\|h\|_\infty < \epsilon$ and $\mathbb{S}_h$ is defined by the map:

$$(1) \quad \mathbb{S}_h : \quad \mathbb{S} \to \mathcal{M} \subset \mathbb{R}^m$$

$$X(z) \mapsto \alpha_{X(z)}(1) = X(z) + h(X(z))\tilde{\Gamma}(X(z)) + O(h^2).$$

The number $\|h\|_\infty$ is called the size of the deformation.
Theorem 1. Assume that $S$ is a compact Riemann surface $C^\infty$—embedded in the orientable Riemannian manifold $\mathcal{M}$ of $\dim \mathcal{M} \geq 3$. Let $S_0$ be any Riemann surface structure on $S$. If there exists a nowhere vanishing smooth section of the normal bundle $N_S$ of $S$ in $\mathcal{M}$, then there exists an embedded $\epsilon$-normal deformation $S_\epsilon$ of $S$, of the form given in (1), which is conformally equivalent to the given Riemann surface $S_0$.

Theorem 2. Theorem 1 is valid if $S$ is a hyperbolic Riemann surface, $C^\infty$–embedded in the orientable Riemannian manifold $\mathcal{M}$ of $\dim \mathcal{M} \geq 3$, of genus $g$ with $m$ punctures and $n$ boundary points and if there exists a nowhere vanishing smooth section of the normal bundle $N_S$ of $S$ in $\mathcal{M}$, where $\mathcal{S}$ is a compactification of $S$.

Theorem 3. Assume that $S$ is any open Riemann surface, $C^\infty$–embedded in the orientable Riemannian manifold $\mathcal{M}$ of $\dim \mathcal{M} \geq 3$. Let $S_0$ be any Riemann surface structure on $S$. If there exists a nowhere vanishing smooth section of the normal bundle of each element in compact exhaustion of $S$ in $\mathcal{M}$, then $S_0$ is conformally equivalent to a complete classical surface in $\mathcal{M}$. A model can be constructed by deforming a given topologically equivalent complete Riemann surface $S$ on each element in compact exhaustion of $S$ via the map (1).

It can be shown that if $\dim \mathcal{M} \neq 4$, then there always exists a nowhere vanishing section of the normal bundle $N_S$ of $S$ in $\mathcal{M}$ if $S$ is compact. When $\dim \mathcal{M} = 4$, the nowhere vanishing section of the normal bundle $N_S$ exists if there are no obstructions. In this case the obstruction lies in the Euler class $e(N_S)$ of the normal bundle $N_S$. That is, if $e(N_S) = 0$, then there is always such a section.

2. Teichmüller Space of Bordered Riemann Surfaces

A Riemann surface $S$ is said to be of finite topological type $(g, m, n)$ if $S$ is biholomorphic to a compact genus $g$ surface $\mathcal{S}$ from which $m$ points and $n(>0)$ hyperbolic disks have been removed.

To define a Teichmüller space for the Riemann surface of the type $(g, m, n)$, let $S_1$ be a topological oriented surface of type $(g, m, n)$. Form a pair $(S_1, f_1)$ consisting of $S_1$ and an orientation-preserving quasiconformal mapping $f_1 : S \to S_1$. It is required that $f_1$ takes the punctures of $S$ into the first set of punctures of $S_1$. The pair $(S_1, f_1)$ is called a marked Riemann surface of the type $(g, m, n)$. $(S_1, f_1)$ and $(S_2, f_2)$ are conformally equivalent if and only if $f_2 \circ f_1^{-1}$ is homotopic to a conformal map of $S_1 \to S_2$. Then we get
Definition 2. The (reduced) Teichmüller space $T^\#(S)$ of a Riemann surface of type $(g, m, n)$ is the set of all conformal equivalence classes of marked surfaces $(S_1, f_1)$ of the type $(g, m, n)$, where $f_1 : S \to S_1$ is an orientation-preserving quasiconformal mapping. We denote the equivalence class $[(S_1, f_1)]$ as $[f_1]$.

Remark. If the boundary of $S = \emptyset$, that is, $S$ is a compact Riemann surface or a conformal type ($n = 0$) Riemann surface, then $T^\#(S)$ is a usual Teichmüller space $T(S)$.

Given any Riemann surface $S$ of type $(g, m, n)$, we define the Teichmüller distance between $[f_1], [f_2] \in T^\#(S)$ by

$$d([f_1], [f_2]) = \inf_h \left\{ \frac{1}{2} \log(\sup_z K_h(z)) \mid h \simeq f_2 \circ f_1^{-1} \right\},$$

where $K_h(z)$, the dilatation of $h$ at $z$, is defined by

$$K_h(z) = \frac{|h_z(z)| + |\bar{h}_z(z)|}{|h_z(z)| - |\bar{h}_z(z)|}$$

and $\simeq$ denotes the free homotopy.

Since the dilatation of a $K$–quasiconformal mapping is invariant under conformal transformations, this distance is well defined.

Then $T^\#(S)$ is a metric space under Teichmüller distance. Further $T^\#(S)$ is a manifold. We now give a local coordinate to $T^\#(S)$ (see Nag [14, p. 151 ff]).

Teichmüller theorem for $S$ of type $(g, m, n)$, $2g - 2 + m + n > 0$, can be deduced from the theorem for finite conformal type (see Abikoff [1]) applied to double $S^d$ of $S$ which is a Riemann surface of type $(2g + n - 1, 2m)$ since we are doubling over the boundary curves. Indeed, any quasiconformal homeomorphism $f : S \to S_1$ extends to a quasiconformal homeomorphism $f^d : S^d \to S_1^d$, of course, by reflection.

Let $\mathbb{H}$ ($\mathbb{L}$, respectively) be upper (lower, respectively) half plane and $G$ be a nonelementary torsion-free (as usual, normalized) Fuchsian group (of the second kind but finitely generated) uniformizing $S = \mathbb{H}/G$; then $S^d = \Omega/G$, where $\Omega = \mathbb{H} \cup \mathbb{L} \cup \{\text{discontinuity region on } \mathbb{R}\}$ is the full region of discontinuity for $G$.

We now introduce the following notation for the space of holomorphic quadratic differentials on $S^d$:

$$Q(S^d) = Q(\Omega) = \{ \omega = \phi dz^2 \mid \phi(\gamma(z))\gamma'(z) = \phi(z), \phi \text{ is holomorphic} \},$$

where $\gamma \in G$ and the norm $\|\omega\|$ of a quadratic differential $\omega$, defined by

$$\|\omega\| = \int \int_{S^d} |\omega|$$

is finite.
Lemma 1. Let $f$ be a quasiconformal map and $\omega$ be a dilatation of the extremal quasiconformal map from $S_\omega$ to $S_{\tilde{\omega}}$ homotopic to the identity on $S$, then we have

\[ (2) \quad ds^2_\omega := \lambda(z) \left( dz + \left\| \frac{\omega(z)}{\lambda(z)} \right\| \right)^2, \]

where $\lambda(z)$ is a smooth real-valued $(1,1)$-form. The metric (2) defines a new conformal structure on $S$, which will be denoted $S_\omega = (S, ds^2_\omega)$.

Lemma 1 (Garsia [5]). If $[f_\omega] \in B_r(\omega_0)$ and if there is a quasiconformal mapping $\chi : S_\omega \to S_{\tilde{\omega}}$, whose dilatation $K_\chi$ satisfies

(i) $K_\chi \leq K_0$, where $K_0$ is a dilatation of the extremal quasiconformal map from $S_\omega$ to $S_{\tilde{\omega}}$ homotopic to the identity on $S$,

(ii) $K_\chi \leq 1 + \delta$ except on $A \subset P$ and

(iii) area $A \leq \eta$,
then there is a constant \( b = b(K_0, \delta, \eta) \) so that
\[
\| \omega' - \omega \| \leq b(K_0, \delta, \eta).
\]
Further, if \( K_0 \) is bounded as \((\delta, \eta) \to (0, 0)\), then \( b(K_0, \delta, \eta) \to 0\).

Proof. See Garsia [5, p. 100 ff].

Next we need the following lemma on the fixed point of the map.

**Lemma 2.** Suppose that a continuous mapping
\[
f : \mathbb{B}_s(p_0) = \{ p \in \mathbb{B}^k \mid |p - p_0| \leq s \} \to \mathbb{B}^k
\]
has the property
\[
|f(p) - p| \leq s \text{ for all } p \in \mathbb{B}_s(p_0)
\]
and for each \( p_0 \in \mathbb{B}^k \), where \( \mathbb{B}^k \) is a \( k \)-dimensional open unit ball. Then
\[
f(p_1) = p_0 \text{ for some } p_1 \in \mathbb{B}_s(p_0).
\]
Proof. See S. Ko [12].

4. Outline of the Proofs

Let \( S \) be any Riemann surface \( \mathcal{C}^\infty \)-embedded in the orientable Riemannian manifold \( \mathcal{M} \) of \( \dim \mathcal{M} \geq 3 \) and \( S_0 \) be any Riemann surface structure on \( S \). We will unify the proofs of all cases.

4.1. Proof of the Theorem 1

Assume that \( S \) and \( S_0 \) are compact Riemann surfaces.

The first part of Theorem 1, the existence of a family of embedded \( \epsilon \)-normal deformations \( \{ S_\epsilon \} \) of \( S \), only requires the existence of a nowhere vanishing section of the normal bundle of \( S \) as we have seen in the Section 1.

To prove the existence of the conformal model, first we fix a map \( h : S \times \mathcal{B}_\epsilon(\omega_0) \to (-\epsilon, \epsilon) \) so that \( h \) is a \( \mathcal{C}^\infty \)-function on \( S \) for each fixed \( \omega \). Let the surface \( S^\omega \) be the \( \epsilon \)-normal deformation of \( S \) defined by the map \( \mathcal{G}^\omega := \mathcal{G}_h(\omega) \) (given in (1)). Denote by \( [\mathcal{G}^\omega] = ([S^\omega, \mathcal{G}^\omega]) \) the conformal equivalence class of the surface \( S^\omega \) as a marked surface \((S^\omega, \mathcal{G}^\omega)\). We then define a map \( \Xi \) of \( \mathcal{B}_\epsilon(\omega_0) \) to \( \mathcal{T}(S) \) by
\[
\Xi : \mathcal{B}_\epsilon(\omega_0) \longrightarrow \mathcal{T}(S) \quad \omega \longmapsto [\mathcal{G}^\omega].
\]

Then, as a consequence of Lemma 2, we will have proved the existence of the conformal model if we can prove that, given \([f_0] = \omega_0 \) and \( \epsilon > 0 \), for \( \omega \) in the
closed ball $\overline{B}_\epsilon(\omega_0) \subset T(S)$, there is a family of deformations $S^\omega$ of $S$ depending on parameters $\omega \in \overline{B}_\epsilon(\omega_0)$ so that the following is true.

**Lemma 3** (Dependence of $S^\omega$ on Parameters $\omega$). In the above notation,

(i) $\Xi : \omega \mapsto [S^\omega]$ is continuous in $\overline{B}_\epsilon(\omega_0)$.

(ii) $||[S^\omega] - [id_\omega]|| \leq \epsilon, \forall \omega \in \overline{B}_\epsilon(\omega_0)$, where $id_\omega : S \to S^\omega$ is the set-theoretic identity map.

Garsia’s Continuity Lemma (Lemma 1) implies that the family $\{S^\omega\}$ satisfies property (i) if the coefficients of $(d\tilde{S}^\omega)^2$ depend continuously on $(z, \omega) \in \tilde{S} \times \overline{B}_\epsilon(\omega_0)$. We give a way of constructions for the functions $h(\cdot, \omega)$ in Section 5. Following those instructions, we may give an explicit formula for $h$; from the formulas it follows directly that this property is satisfied.

To prove property (ii), we let $\chi = S^\omega \circ (id_\omega)^{-1} : S^\omega \to S^\omega$, then its dilatation $K_\chi$ satisfies

$$K_\chi^2 = \sup_{\text{inf}} \frac{(d\tilde{S}_\omega)^2}{(d\tilde{S}_0)^2},$$

where both the supremum and infimum are taken over all directions and $ds_\omega^2$ is as defined in (2). We can construct the set $A$ and determine the constants $\delta$, $\eta$ and the constant $\epsilon$ as we construct $h$, and then we can compute $K_\chi$ using them. For all of which we get $b(K_\chi, \delta, \eta) \leq \epsilon$ in Garsia’s Continuity Lemma 1 (See S. Ko [7, 8, 12]). Then application of Garsia’s Continuity Lemma 1 gives property (ii).

**Continuation of the outline of the proof of the existence of the conformal model**  By Lemma 3, the function $\Xi$ satisfies the hypotheses of Lemma 2. Therefore there is a point $\omega_1 \in \overline{B}_\epsilon(\omega_0)$ so that

$$\Xi(\omega_1) = [S^{\omega_1}] = \omega_0 = [f_0], \text{ where } f_0 : S \to S_0,$$

i.e., for this $\omega_1 \in \overline{B}_\epsilon(\omega_0)$, the deformed surface $S^{\omega_1}$ can be mapped conformally onto $S_0$ by a mapping homotopic to $f_0 \circ (S^\omega)^{-1}$.

Finally in Section 6, we collect all the facts needed to finish the proof of Theorem 1.

### 4.2. The Proof of the Theorem 2

In the following special cases among Riemann surfaces of type $(g, m, n)$, surfaces of the same type are always conformally equivalent;

(i) $(0, i, 0)$, $i = 0, 1, 2, 3$ : $i$–components punctured sphere.
(ii) \((0, i, 1)\), \(i = 0, 1\) : \(i\)-components punctured disk.

(iii) \((0, 0, 2)\).

Therefore, for the above cases, the Theorem 2 is already proved. Henceforth we omit the above cases.

So let \(S\) be a subsurface (of type \((g, m, n)\)) of a compact Riemann surface in \(\mathcal{M}\), its boundary consists of \(m\) isolated points \(p_i (i = 1, \cdots, m)\) and \(n\) analytic boundary curves \(\gamma_j (j = 1, \cdots, n)\), where \(2g - 2 + m + n > 0\) so that \(\tilde{S} = \Delta\). (In this case, we call \(S\) a hyperbolic Riemann surface.) Let \(S_0\) be any Riemann surface structure on \(S\).

Then, we get the double \(S^d\) of \(S\) and construct a doubly sheeted covering \(\hat{S}^d\) of \(S^d\) and give fundamental domain \(P\) in \(\Delta\) to \(\hat{S}^d\).

Yet we make the following agreement, because of the homogeneity of the Riemann surface, that through possible distinction we move \(m\) distinguished points in one half of \(S^d\) into the other half so that one half of \(S^d\) has \(2m\) distinguished points and the other none. We may then think that \(\hat{S}^d\) originated from \(S^d\) which is of type \((2g + n - 1, 2m)\) : the distinguished point \(p_i\) are linked on \(S^d\) by curves and those curves are mutually disjoint and disjoint from \(\gamma_j\) so that each \(p_i\) is either beginning or ending point. Two copies \((S^d)_i, i = 1, 2,\) of \(S^d\), are cut along these curves and they are glued crosswise together along these curves. Let \(\Lambda\) be the union of the curves on \(P\) which corresponds to these cuts and \(\gamma_j\). Let \(O'\) and \(O\) be open neighborhoods of \(\Lambda\) with the following properties;

(a) \(\overline{O} \subset O'\).
(b) area \(O' < \frac{1}{2}\eta\).
(c) \(P - O\) corresponds to the compact subdomain of \(\hat{S}^d\) which decomposes into 4 connected components (2 if \(n = 0\), each of them correspond to compactification \(\overline{S}\) of \(S\).

To this \(\overline{S}\), the compactification of \(S\), we characterize \(\epsilon\) on \(\overline{S}\). And only on this \(\overline{S}\), \(S\) will be deformed. We identify the surface \(\overline{S}\) with the subdomain \(P_1\) of \(P\).

In the Section 5, we give conditions for a deformation function \(h\) on \(P_1\). We change \(h\) so that it vanishes in the set \(O \cap P_1\) and stays unchanged outside the set \(O' \cap P_1\).

Once we have got the deformation function \(h(\cdot, \omega)\) on \(P_1\), then we have the deformed surface \(\overline{S}'\) defined like in (1). The natural structure of the deformed surface \(\overline{S}'\) will be transplanted and it will be extended onto whole \(\hat{S}^d\) by means of anticonformal mapping of \(\Delta\) onto itself and suitable transformation from \(G\), the
cover group of $\Delta$ over $S^d$, whereby all symmetric properties of $\hat{S}^d$ will be transferred onto the deformed surface $\hat{S}^{d\omega}$.

Because of the symmetry of the surface $\hat{S}^{d\omega}$, its module belongs to the space $T^#(S) \simeq Q_1^{\text{sym}}(S^d) \simeq B^{6g-6+2n+3n}$ (open unit ball).

The existence of the families of deformations This only needs a characterization of $\epsilon$ on $\hat{S}$. But it is an easy consequence of the definition of exponential map and the property of the map $\beta$. This only requires the existence of a nowhere vanishing section of the normal bundle of $\hat{S}$.

The existence of the conformal model Let, for convenience, $\phi_0 : \hat{S}^d \rightarrow \hat{S}_0^d$ be a homeomorphism such that $\omega_0 = [\phi_0] \in T(\hat{S}^d)$ is a base point. Now we apply the same procedure as in the Section 4.1 to $\phi_0, \hat{S}^d, \hat{S}_0^d$ and $T^#(S)$ in place of $f_0, S, S_0$ and $T(S)$, we obtain the following conformal mapping $\hat{g}$,

$$\hat{g} : \hat{S}^{d\omega} \rightarrow \hat{S}_0^d,$$

that is, the proof of the existence of the surface $\hat{S}^{d\omega}$ for some $\omega_1 \in T^#(S)$ in the neighborhood of $\omega_0$. This mapping $\hat{g}$ induces a conformal mapping $g$ (see Ahlfors [2]),

$$g : S^{\omega_1} \rightarrow S_0.$$

Therefore the resulting surface $S^{\omega_1}$ from $S$ via deformation in the direction of the normal is the desired conformal model of the given surface $S_0$. This completes the proof of the theorem for the bordered surfaces if we construct a deformation function $h$ for compact surface $\hat{S}$.

4.3. The Proof of the Theorem 3

4.3.1. Idea of the Proof. Assume that $S$ and $S_0$ are non-compact Riemann surfaces. Then there exists a topological mapping $f : S_0 \rightarrow S$ by a consequence of the choice of $S$ and $S_0$. In terms of exhaustions, we do the following constructions. Since every Riemann surface admits a countable compact exhaustion by a subsurface, we may choose a regular exhaustion, that is, a sequence $\{S_i^0\}$ on $S_0$, of relatively compact regular subregions, such that $S_i^0 \subset S_i^{i+1}$, $\cup S_i^0 = S_0$ and $\partial S_i^0$ consists of analytic arcs.

It is easy to show that $S_i^0$ can be mapped by $f_i$ topologically on a classical surface $S_i$ such that $\partial S_i$ consists of circles contained in $\partial B_i$, where, for $n = \dim \mathfrak{M}$,

$$B_i = \left\{ (x_1, x_2, \ldots, x_n) \left| \sum_{j=1}^{n} x_j^2 \leq i^2 \right. \right\}, \ S_i \subset B_i, \ S_{i+1} \cap B_i = \overline{S_i}.$$
with dim $B_i = 2 = \dim S = \dim S_0$, and $f_{i+1}|_{S_0^i} = f_i$, $f = \lim f_i$ and $S = \cup S_i$ satisfy the above conditions ([16]).

We may assume that $S_0^1$ is a disk. Let $p \in S_0^1$ and $q \in \partial S_0^1$ be distinguished points and put $p' = f(p)$, $q' = f(q)$ and $f(S_0^1) = S$. If $S_0$ is simply connected, we introduce 4 distinguished points.

We will deform $S$ in successive steps such that the $i$--th deformation ($i \geq 2$) takes place on $S_i - S_{i-1}$ only, and we will denote the result of the $i$--th deformation by $S'$. Let $S_i'$ be the part of $S'$ corresponding to $S_i$. We will show that $S_0^i$ can be mapped conformally onto $S_i'$ by a mapping $f_i$ with the additional properties $f_i(p) = p'$, $f_i(q) = q'$, $i \geq 1$. The existence of $f_1$ follows by Riemann’s mapping theorem, the existence of $f_i$, $i \geq 2$, will be proved by induction.

If this is accomplished, our theorem is implied by the following

**Lemma 4.** Let $\{S_0^i\}$ and $\{S_i\}$ be exhaustions of the noncompact Riemann surfaces $S_0$ and $S$, and let $p$ and $q$ be fixed points in $S_0$, $p'$ and $q'$ be fixed points in $S$. If the mappings $f_i : S_0^i \to S_i$ are conformal and if $f_i(p) = p'$, $f_i(q) = q'$, $i \geq i_0$, then $S_0$ and $S$ are conformally equivalent.

**Proof.** See S. Ko [12].

4.3.2. The existence of the functions $f_i$. Suppose $S$ is any open Riemann surface, $C^\infty$-embedded in the orientable Riemannian manifold $M$ of dim $M \geq 3$. Let $S_0$ be any Riemann surface structure on $S$. Let $\{S_0^i\}$ and $\{S_i\}$ be exhaustions of $S_0$ and $S$ respectively. Assume that $S_{i-1}$ is deformed into a surface $S_{i-1}'$ such that conformal map $f_{i-1} : S_0^{i-1} \to S_{i-1}'$ with $f_{i-1}(p) = p'$ and $f_{i-1}(q) = q'$ exists. We are going to construct $S_i'$ and $f_i$.

Let $S_i' = (S_i - S_{i-1}) \cup S_{i-1}'$. Fix a global coordinate $z \in S_i''$ such that $X : S_i'' \to S_i''$ is a conformal parametrization of $S_i''$. Then the Riemann surface structure of $S_i''$ may be viewed as induced by the metric $(dX)^2 = \lambda^2(z)|dz|^2$, where $\lambda(z)$ is a smooth real valued $(1,1)$--form.

Extend $f_{i-1}$ to $S_i''$ such that the extended map $g : S_0'' \to S_i'' \subset \mathbb{R}^m$ is $K$--quasiconformal for a suitable $K$, $C^\infty$ except perhaps on $\partial S_0^{i-1}$ and such that, for the complex dilatation $\mu(g) = \frac{\partial g}{\partial z}$ of $g$,

\[\mu(g) = 0 \quad \text{on} \quad S_0^{i-1}, \quad \mu(g) = \frac{i}{2} \quad \text{on a disk} \quad D \quad \text{in} \quad S_0^i - S_0^{i-1}.\]

Such an extension is certainly possible (see Lehto [13, p. 89]).
By the previous constructions, \( S_0^{-1} \rightarrow S_{i-1} \) is conformal. By the previous paragraph and the properties of the quasiconformal mappings, \( g : S_0 \rightarrow S_i^{''} \) is a homeomorphism and so it has an inverse \( \phi_0 : S_i^{''} \rightarrow S_0 \) which is quasiconformal. We use the Teichmüller space \( T^\#(S_i^{''}) \), and let \( [\phi_0] = \omega_0 \in T^\#(S_i^{''}) \).

For later use, we define a surface \((S_i^{''})_\omega\) as follows: For any \( \omega = \phi_\omega(z)dz^2 \in T^\#(S_i^{''}) \setminus \{0\} \), define a metric \( ds_{\omega}^2 \) by

\[
ds_{\omega}^2 := \lambda^2(z) |dz + \Psi_\omega(z)d\bar{z}|^2,
\]

where \( \lambda \) is a smooth real-valued \((1,1)\)-form and

\[
\Psi_\omega(z) = \begin{cases} 
\frac{\|z\|}{\|z(\omega)\|} & \text{on } g(D) \\
0 & \text{outside } g(D).
\end{cases}
\]

We call a surface \((S_i^{''}, ds_{\omega}^2)\) a surface \((S_i^{''})_\omega\).

Now we fix a map

\[
h : S_i^{''} \times \overline{B}_r(\omega_0) \rightarrow (-\epsilon, \epsilon)
\]

so that \( h \) is a \( C^\infty \)-function with support on \( S_i - S_{i-1} \) for each fixed \( \omega \). Conditions of the \( h \) will be given explicitly in Section 5.

Now apply the same procedure as in the Section 4.1 to \( S_i^{''}, S_0^{''}, T^\#(S_i^{''}) \) and \( \phi_0 \) in place of \( S, S_0, T(S) \) and \( f_0 \), we obtain that there exists deformed surface \((S_i^{''})_{\omega_1} = S_i'\)

and a conformal map \( f_i : S_0' \rightarrow S_i' \) satisfying \( f_i(p) = p', f_i(q) = q' \).

To use in the next section, we introduce \( P \) as a fundamental domain for \( S_i^{''} \) in \( S_i'' \)

and \( P_1 \) (resp. \( P_{i-1} \)) for \( S_i \) (resp. \( S_{i-1} \)). Then the fundamental domain for \( S_i - S_{i-1} \)

will be the domain \( P_1 - P_{i-1} \).

5. The Deformation Function \( h \)

Let \( R \) represent \( S \) (for a compact case), or \( \overline{S} \) (for a bordered case), or \( S_i^{''} \)

(for an open case). Let \( \mathcal{G}_{\omega}^{''} \) be the map \( \mathcal{G}_{h(\omega)} \) defining the deformed surface \( R_{\omega}^{''} \) as in formula (1). Then to complete the Embedding Theorems, we have to compare two metrics \((d\mathcal{G}_{\omega}^{''})^2 \) and \( ds_{\omega}^2 \) so that we compute \( K_\chi \) given in the form (3). In the course of this comparison, we obtain the conditions for the deformation function \( h : R \rightarrow (-\epsilon, \epsilon) \). This deformation function \( h : R \rightarrow (-\epsilon, \epsilon) \) has to satisfy the following properties:

(a) \( h \) is \( C^\infty \).

(b) \( ||h||_\infty < \epsilon \).

(c) \( (dh)^2 \) is proportional to \((\alpha_\omega dx + \beta_\omega dy)^2\), where \( \alpha_\omega \) and \( \beta_\omega \) are some functions of \( \omega \) and \( \Psi_\omega(z) \) given explicitly in the comparison of metrics.
We would like to define a function $h$ satisfying condition (c) except on a sufficiently small set. But $dh$ remains bounded on this set. Condition (c) suggests that we express $(dh)^2$ in terms of $\alpha_\omega$ and $\beta_\omega$.

For a compact Riemann surface of genus 1, we are tempted to write $h = \alpha_\omega x + \beta_\omega y$, but that function will usually violate condition (b). Therefore we must abandon global linearity and get a smooth approximation to a piecewise linear function $h$. We wish to apply Garsia’s Continuity Lemma 1. The hypotheses of the Lemma would be simple if we could apply the saw-tooth function with slope $\pm 1$ to $\alpha_\omega x + \beta_\omega y$. The sign change doesn’t affect condition (c) except at the corners and this can be smoothed away on a small set. On the other hand, near the tip, condition (a) is violated. A smoothing procedure gives the necessary improvements. But then condition (c) is again broken. Again we smooth away the problem on a small set.

On all other general Riemann surfaces, $\alpha_\omega$ and $\beta_\omega$ must be non-constant functions of $z$. The definition of $h$ will come as a solution of a differential equation in which $\alpha_\omega$, $\beta_\omega$ and their derivatives appear as coefficients. In order to get a $C^\infty$ solution, we need $\alpha_\omega$, $\beta_\omega$ to be smooth on all of the fundamental domain $P$ (for compact or open case) or $P_i$ (for bordered case), that is on $R$. Also they, together with their derivatives, must change as little as possible. For $h$ to be well-defined on $R$, it is convenient that it be zero, in a neighborhood of the edges of $P$ (for compact) and $P_i$ (for a bordered case), or on $P_{i-1}$ and in a neighborhood of the edges of $P_i - P_{i-1}$ (for an open case), but remains smooth.

The construction processes and the exact forms of $h$ are similar but the details are little different for each case, and we do not attempt to do that here, for it is a long and complicated way to go. Instead, for full details of the constructions of $h$, we refer to S. Ko [7, 8, 11, 12].

6. Final Words

Once we have got the function $h$, then we can compute $K_\chi$. Also in the course, we got the small set $A$. Therefore if we take $\epsilon = \frac{1}{2}\min\{1 - \|\omega_0\|, \|\omega_0\|\}$, then with $B_\epsilon(\omega_0) \subset T^\#(R) \setminus \{0\}$, Garsia’s continuity lemma 1 is satisfied, and so we may now complete the process in the Section 4. (Here $R$ has the same meaning as in the Section 5.)

References


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