EMBEDDING COMPACT RIEMANN SURFACES IN RIEMANNIAN MANIFOLDS

SEOKKU KO
COMMUNICATED BY ROBERT M. HARDT

ABSTRACT. Any compact Riemann surface has a conformal model in any orientable Riemannian manifold. Precisely, we will prove that, given any compact Riemann surface, there is a conformally equivalent model in a pre-specified orientable Riemannian manifold. The techniques we use include Garsia's Continuity Lemma, Brouwer's Fixed Point Theorem along with techniques from Teichmüller theory.

1. Introduction

$C^\infty$-embedded surfaces are called classical surfaces if they are viewed as Riemann surfaces whose conformal structure is given in the following natural way: the local coordinates are those which preserve angles and orientation. The existence of local coordinates is highly non-trivial, it means solving the Beltrami differential equation. This was done for analytic embeddings by Gauss, for differential embeddings by Korn-Lichtenstein. Because of the fundamental importance of this problem in the theory of quasiconformal mappings, it was investigated more thoroughly in the 1950's. For the most elegant treatment see [6].

In his lectures, Felix Klein emphasized that classical surfaces should be viewed as Riemann surfaces, i.e., as domains of analytic functions and integrals. In 1882, Klein posed the question of whether every Riemann surface is conformally equivalent to a classical surface (see Klein [23, p. 635]).

The earliest results in this direction were that every compact surface of genus zero is conformally equivalent to the sphere, every non-compact planar classical surface is equivalent to a subregion of the plane, and a compact classical surface
of genus 1 is conformally equivalent to a torus of revolution provided its modulus is purely imaginary.

The first result beyond these facts was obtained by Teichmüller ([41]). He deformed an embedded surface by moving each point in the normal direction and studied the dependence of the conformal structures of the perturbed surface on deformation parameters.

Around 1960, A. Garsia ([17]) proved that every compact Riemann surface can be conformally immersed in the Euclidean 3-space $\mathbb{R}^3$. He stated that he had found a realization of every compact Riemann surface as a classical surface although Klein required that classical surfaces be embedded. Garsia’s proof uses Teichmüller’s idea, results, and constructions inspired by Nash’s embedding theorem and Brouwer’s fixed point theorem.

In 1970, Rüedy extended Garsia’s result to open Riemann surfaces $\mathcal{S}$ by applying Garsia’s techniques to compact exhaustions of $\mathcal{S}$ ([36]). He also proved that every compact Riemann surface can be conformally embedded in $\mathbb{R}^3$ ([37, 38]).

2. The Embedding Problem

Every smooth compact orientable surface in a Riemannian manifold is naturally a Riemann surface. We may use the extrinsic metric, that is the metric induced by the surrounding Riemannian manifold, to introduce a conformal structure. In fact, every orientable Riemannian 2-manifold has the natural structure of a Riemann surface in which the angles are the same measured using $ds^2$ or using the holomorphic local coordinates. As Gauss understood, and proved existence in the case of real analytic surfaces in $\mathbb{R}^3$, the passage from the Riemannian structure on a surface to a Riemann surface structure is precisely the introduction of isothermal coordinates.

We may ask whether we can get a conformally equivalent model in a Riemannian manifold for any given compact Riemann surface $\mathcal{S}_0$. To give the answer we will study the deformation of surfaces embedded in an orientable Riemannian manifold.

Our main result here is the extension of the Garsia-Rüedy theorem to compact Riemann surfaces in orientable Riemannian manifolds.

2.1. The main results. Let $\mathcal{M}$ be an orientable Riemannian manifold of dim $\mathcal{M} \geq 3$ and let $\mathcal{S}$ be a closed $\mathcal{C}^\infty$ - embedded Riemann surface in $\mathcal{M}$. We will assume that the genus $g$ of a compact Riemann surface $\mathcal{S}$ always is $\geq 1$ unless otherwise specified. We briefly examine one method of constructing deformations
of \( S \) in \( \mathcal{M} \). Throughout the paper, we will use the following standard symbols. \( \mathbb{C} \) is the complex numbers, \( \mathbb{R}^m \) is the \( m \)-dimensional Euclidean space, \( \mathbb{B}^n \) is the open unit ball in \( \mathbb{R}^n \), and \( \Delta \) is the open unit disk in the plane.

Let \( \Gamma : S \hookrightarrow NS \setminus \Gamma_0 \) be a nowhere vanishing smooth section (with unit length) of the normal bundle \( NS \) of \( S \) in \( \mathcal{M} \). Such sections may not always exist. (See Section 2.2 for details.) Let \( h : S \to (-\epsilon, \epsilon) \) be a \( \mathcal{C}^\infty \)-function on \( S \) and call \( \{ h(x)\Gamma(x) \} \) a normal vector field on \( S \).

Let \( \mathcal{M}_1 \) be the subset of \( NS \) consisting of all pairs \( (x, r) := (x, r\Gamma(x)) \) for all \( x \in S \), where \( r \) is any number satisfying \( |r| < 2\epsilon \). Then \( \mathcal{M}_1 \) contains the pair \( (x, h(x)\Gamma(x)) \). Also let \( \mathcal{M}_2 \) be the set of all points

\[
\{ y \in \mathcal{M} : y = \exp r\Gamma(x), r \in (-2\epsilon, 2\epsilon), (x, r) \in \mathcal{M}_1 \},
\]

then \( \mathcal{M}_2 \) is a Riemannian submanifold of \( \mathcal{M} \) for \( \epsilon \) sufficiently small. Again, for sufficiently small \( \epsilon \), the map \( \beta : \mathcal{M}_1 \to \mathcal{M}_2 \), defined by the exponential map \( \beta(x, r) = \exp r\Gamma(x) \), is a diffeomorphism. This diffeomorphism depends on either \( S \) being compact or on \( \mathcal{M} \) having some uniform bounds on its local geometry.

By Nash’s Embedding Theorem, there is a \( \mathcal{C}^\infty \)-isometric embedding \( j : \mathcal{M} \hookrightarrow \mathbb{R}^m \) for some sufficiently large \( m \). This embedding allows us to consider \( S \) and \( \mathcal{M} \) as subsets of \( \mathbb{R}^m \).

Assume that \( \tilde{S} \) is the holomorphic universal covering of \( S \). Let \( X : \tilde{S} \to \mathcal{M} \subset \mathbb{R}^m \) be a local conformal parametrization of \( S \) in the orientable Riemannian manifold \( \mathcal{M} \subset \mathbb{R}^m \).

For \( X(z) \in S \), let

\[
\alpha_{X(z)} : (-2, 2) \to \mathcal{M} \subset \mathbb{R}^m \quad t \mapsto \beta(X(z), th(X(z))) := \exp th(X(z))\Gamma(X(z)).
\]

Then \( \alpha_{X(z)}(t) \) is a \( \mathcal{C}^\infty \)-curve for which \( \alpha_{X(z)}(0) = X(z) \) and

\[
\alpha_{X(z)}(1) = \beta(X(z), h(X(z))) = \exp h(X(z))\Gamma(X(z)).
\]

Let \( \tilde{\Gamma}(X(z)) \in T_{X(z)}\mathbb{R}^m \) be a unit tangent vector in \( \mathbb{R}^m \) to the curve \( \alpha_{X(z)}(t) \) at the point \( X(z) \in S \). We then have

\[
\alpha_{X(z)}(1) = X(z) + h(X(z))\tilde{\Gamma}(X(z)) + O(h^2(X(z))), \quad |h(X(z))| < \epsilon.
\]

This is precisely the statement that \( h(X(z))\tilde{\Gamma}(X(z)) + O(h^2(X(z))) \) is the tangent vector in \( \mathbb{R}^n \) to the curve \( \alpha_{X(z)}(t) \) at the point \( \alpha_{X(z)}(0) = X(z) \).

Now, for any given sufficiently small \( \epsilon > 0 \), we may define a normal deformation of \( S \).
**Definition 1.** In the above notation, a surface $S_h : \mathcal{M} \rightarrow M$ is called an $\epsilon$–normal deformation of $S$ if, for a given small $\epsilon > 0$, $h$ is a $C^\infty$–real-valued function on $S$ such that $\|h\|_\infty < \epsilon$ and $S_h$ is defined by the map:

$$S_h : \mathcal{M} \rightarrow M \subset \mathbb{R}^m$$

$$X(z) \mapsto X(z) + h(X(z))\Gamma(X(z)) + O(h^2) \in M.$$

The number $\|h\|_\infty$ is called the size of the deformation.

We now may state the Embedding Theorem as follows.

**Theorem 2.1 (Main Theorem).** Assume that $S$ is a compact Riemann surface $C^\infty$–embedded in the orientable Riemannian manifold $M$ of dim $M \geq 3$. Let $S_0$ be any Riemann surface structure on $S$. If there exists a nowhere vanishing smooth section of the normal bundle $N S$ of $S$ in $M$, then

1. **[Existence of normal deformations of $S$]** There exists an $\epsilon = \epsilon(S)$ so that there is an embedded $\epsilon$–normal deformation $S_h$ of $S$, of the form given in (2.2).

2. **[Existence of Conformal Models]** There exists an $\epsilon$-normal deformation $S_h$ of $S$ which is conformally equivalent to the given Riemann surface $S_0$.

A guide to the proof is given in Section 5.2. The details of arguments and computations are given in Section 6.

Let $\mathcal{E}$ be the set of $\epsilon$-normal deformations of $S$ in $\mathcal{M} \subset \mathbb{R}^m$ which are $C^\infty$-embedded. We may parametrize the family $\mathcal{E}$ by setting

$$\text{Maps}_\epsilon(S) = \{ f : S \rightarrow f(S) \subset \mathbb{R}^m \mid \text{defines an } \epsilon \text{– normal deformation of } S \},$$

then for each $f \in \text{Maps}_\epsilon(S)$, there is exactly one $\epsilon$-normal deformation $f(S) \in \mathcal{E}$ corresponding to this $f$. Two maps $f_1$ and $f_2$ in $\text{Maps}_\epsilon(S)$ are called equivalent if $f_2 \circ f_1^{-1}$ is homotopic to a conformal mapping of $f_1(S)$ onto $f_2(S)$. Denote by $[f]$ the equivalence class of $f$. (see Section 5.3 too for the topology of $\text{Maps}_\epsilon(S)$).

We now recall the definition of the Teichmüller space $T(S)$. Choose a fixed compact Riemann surface $S$ of genus $g$ and form the pair $(S_1, f_1)$ consisting of a Riemann surface $S_1$ of genus $g$ and an orientation-preserving quasiconformal mapping $f_1 : S \rightarrow S_1$. The pair $(S_1, f_1)$ is called a marked Riemann surface.

Two marked Riemann surfaces $(S_1, f_1)$ and $(S_2, f_2)$ are said to be conformally equivalent iff $f_2 \circ f_1^{-1}$ is homotopic to a conformal mapping of $S_1 = f_1(S)$ onto $S_2 = f_2(S)$. 


A homeomorphism \( f : S_1 \to S_2 \) is called a mapping of \((S_1, f_1)\) onto \((S_2, f_2)\) if \( f \) is homotopic to \( f_2 \circ f_1^{-1} \).

**Definition 2.** The Teichmüller space \( T(S) \) is the set of all conformal equivalence classes of marked Riemann surfaces \((S_1, f_1)\), where \( f_1 : S \to S_1 \). We will also denote the equivalence class \([S_1, f_1]\) as \([f_1]\).

We then have the following characterization of maps of \( \text{Maps}_s(S) \) into the Teichmüller space \( T(S) \) (see Section 3 for the topology of \( T(S) \)).

**Theorem 2.2.** Let \( S \) be a compact Riemann surface. Then the map of the space \( \text{Maps}_s(S) \) into the Teichmüller space \( T(S) \), defined by \( f \mapsto [f] \), is continuous in the \( C^1 \)-topology. This map is not continuous in the \( C^0 \) (uniform)-topology.

The proof is given in Section 5.3.

### 2.2. Additional remarks.

It can be shown that if \( \dim M \neq 4 \), then there always exists a nowhere vanishing section of the normal bundle \( NS \) of \( S \) in \( M \). When \( \dim M = 4 \), the nowhere vanishing section of the normal bundle \( NS \) exists if there are no obstructions. In this case the obstruction lies in the Euler class \( e(NS) \) of the normal bundle \( NS \) of the surface \( S \subset M \). That is, if \( e(NS) = 0 \), then there is always such a section. In Section 5.1, we will sketch the proof of the existence of such a section. If \( e(NS) \neq 0 \), then we know that there are sections with isolated zeroes.

Even in dimension 4, we may be able to construct conformal models if we can get rid of the obstruction by constructing suitable deformation function \( h \). However this will not be treated here-rather it will be included in a future study.

### 3. A Coordinate System in the Teichmüller Space

In the early 1940’s, O. Teichmüller proved two theorems which now form the foundation of the deformation theory of Riemann surfaces. They are known as Teichmüller’s Existence and Uniqueness Theorems or, collectively, as Teichmüller’s Theorem. Even though Teichmüller had complete proofs of his theorems, the first proof of the theorem which was acceptable (easier) to the mathematics community was given by Ahlfors in 1954 ([4]). Teichmüller’s proof of the uniqueness part is still the most elementary. The easiest existence proof is now based on Bers \( \mu \)-trick, which we will use in this section to define a coordinate system on the Teichmüller space of a surface.

Given any Riemann surface \( S \) of any genus \( g \geq 1 \), we define the Teichmüller distance between \([f_1]\), \([f_2]\) \( \in T(S) \) by
\( d([f_1], [f_2]) = \inf \left\{ \frac{1}{2} \log(\sup_z K_h(z)) \mid h \simeq f_2 \circ f_1^{-1} \right\}, \)

where \( K_h(z) \), the dilatation of \( h \) at \( z \), is defined by

\[
K_h(z) = \frac{\sup \left\| \frac{dh}{dz} \right\|}{\inf \left\| \frac{dh}{dz} \right\|} = \frac{|h_z(z)| + |h_{\bar{z}}(z)|}{|h_z(z)| - |h_{\bar{z}}(z)|},
\]

where \( \sup \) and \( \inf \) are taken over all directions at \( z \), and \( \simeq \) denotes free homotopy.

Since the dilatation of a \( K \)-quasiconformal mapping is invariant under conformal transformations, this distance is well defined.

**Theorem 3.1.** A Teichmüller space \( T(S) \) is a metric space with the Teichmüller metric.

**Proof.** See Abikoff [1], Bers [8], Gardiner [13] or Nag [30]. \( \square \)

Moreover, \( T(S) \) is a manifold. First assume that the genus \( g \) of the surface \( S \) is \( g \geq 2 \). Then local coordinates may be defined using the space \( M(S) \) of Beltrami differentials on \( S \) and holomorphic quadratic differentials \( Q(S) \). From an elementary corollary of the Riemann-Roch Theorem, we know that \( Q(S) \) is a \( 3g - 3 \)-dimensional complex vector space. We set

\[
M(S) = \left\{ \mu \mid \mu(\gamma(z))\gamma'(z) = \mu(z), \ |\mu|_\infty < 1, \ z \in \Delta, \ \gamma \in G \right\}
\]

\[
Q(S) = \left\{ \omega = \phi dz^2 \mid \phi(\gamma(z))\gamma'(z)^2 = \phi(z), \ |\omega| = \int_\Delta |\omega| < \infty, \ \gamma \in G \right\}
\]

where \( G \) is the universal covering group for \( S \) acting on the unit disk \( \Delta \).

For \( \mu \in M(S) \), let \( w^\mu(z) \) denote the uniquely determined normalized solution of the Beltrami equation \( w_z = \mu(z)w_z \) that maps \( \Delta \) homeomorphically onto itself. \( w^\mu \) covers a homeomorphism \( f^\mu : S \rightarrow S^\mu = \Delta/(w^\mu G w^\mu^{-1}) \). Thus it defines a new marked Riemann surface \((S^\mu, f^\mu)\). Since \( w^\mu \) is orientation-preserving, so is \( f^\mu \). We have defined a mapping

\[
\Phi : M(S) \rightarrow T(S) \quad \mu \mapsto \left[ (S^\mu, f^\mu) \right].
\]

Note that

\[
K[f^\mu] = \frac{1 + |\mu|_\infty}{1 - |\mu|_\infty}.
\]
Lemma 3.2. For any $\omega = \phi dz^2 \in Q(S) \setminus \{0\}$ and any $k \in [0, 1)$ the associated form $\mu(k, \phi) = k \frac{\phi}{|\phi|}$ is an element of $M(S)$ with $|\mu|$ essentially constant on $S$ at the value $\|\mu\|_{\infty} = k$. Moreover, $\mu(k, \phi) = \mu(k', \phi')$ a.e. if and only if $k = k'$ and $\frac{\phi}{\phi'} \in \mathbb{R}^+$. 

Proof. See Nag [30, p. 147]. □

Let $Q_1(S)$ be the open unit ball in $Q(S)$. For $\omega \in Q_1(S)$, let $\Psi(\omega) = \mu(\|\omega\|, \omega) \in M(S)$, where

$$\mu(\|\omega\|, \omega) = \begin{cases} 0 & \text{if } \omega = 0 \\ \frac{|\phi(z)|}{|\phi(z)|} \|\omega\| & \text{if } \omega \neq 0. \end{cases}$$

Note that $\|\Psi(\omega)\|_{\infty} = \|\mu(\|\omega\|, \omega)\|_{\infty} = \|\omega\|$. Then by Lemma 3.2 and Teichmüller’s Theorem we get

Proposition 3.3. $\Omega = \Phi \circ \Psi : Q_1(S) \to T(S)$ given by $\Omega(\omega) = \Phi(\mu(\|\omega\|, \omega))$ is a homeomorphism.

Proof. See Abikoff [1] or Nag [30]. □

We have omitted the case genus $g = 1$. It is classical that the marked tori are holomorphically parametrized by the points in the upper half-plane $\mathbb{H}$ or in $\Delta$. We have shown

Theorem 3.4. Suppose $S$ is a compact Riemann surface of genus $g$. If $g > 1$, then the Teichmüller space $T(S)$ embeds as the open unit ball $Q_1(S)$ in a normed vector space $Q(S)$ of real dimensions $6g - 6$. If $g = 1$, then $T(S)$ is holomorphically equivalent to $\Delta$.


Therefore $\omega \in Q_1(S)$ gives a global coordinate for $T(S)$ if $g \geq 2$. (For more information, see Abikoff [1, p. 1-36], Ahlfors [4, p. 54-58] or Nag [30, p. 131-188].)

If $S$ has genus 1, then the holomorphic universal cover $\tilde{S}$ of $S$ is holomorphically equivalent to $\mathbb{C}$. Let $L$ be the universal covering group. $\tilde{\omega} = dz^2$ is a holomorphic quadratic differential on $\mathbb{C}$ which is invariant under $L$. Thus $dz^2$ covers a holomorphic quadratic differential on $S$. Any other holomorphic quadratic differential on $\mathbb{C}$ is given by $f(z)dz^2$, where $f(z)$ is entire. The only entire functions which are $L$–invariant are constants. But any holomorphic quadratic differential on $S$ lifts to a holomorphic quadratic differential on $\mathbb{C}$. Thus $Q(S)$ consists of the constant multiples of the projection $\omega_1$ of $dz^2$. Thus Teichmüller space $T(S)$ of $S$ may again be identified with $Q_1(S)$. 
For any \( \omega = \phi_\omega(z)dz \in Q_1(S) \setminus \{0\} \), define a metric on \( S \) by
\[
ds^2_\omega := \lambda^2(z) \left| dz + \| \omega \| \frac{\phi_\omega(z)}{|\phi_\omega(z)|} dz \right|^2,
\]
where \( \lambda^2 > 0 \) is a smooth real-valued \((1,1)\)-form. The metric (3.3) defines a new conformal structure on \( S \), which will be denoted \( S_\omega = (S, \frac{\omega}{\| \omega \|}, \| \omega \|) \).

Suppose we have two metrics \( ds^2_{\omega_1} \) and \( ds^2_{\omega_2} \) on \( S \). Then the identity map on \( S \) induces a quasiconformal mapping \( f \) between \( S_{\omega_1} \) and \( S_{\omega_2} \). Let \( w_1 \) and \( w_2 \) be conformal local coordinates on \( S_{\omega_1} \) and \( S_{\omega_2} \) respectively. Then, since
\[
|dw_1|^2 = ds^2_{\omega_1} \quad \text{and} \quad |dw_2|^2 = ds^2_{\omega_2}
\]
by the conformality of the local coordinates \( w_1 \) and \( w_2 \) with respect to \( ds^2_{\omega_1} \) and \( ds^2_{\omega_2} \) respectively, we can write the dilatation of \( f \) in terms of these metrics by
\[
K_f(w_1) = \sqrt{\sup ds^2_{\omega_2} / \inf ds^2_{\omega_1}},
\]
where supremum and infimum are taken over all directions at \( w_1 \).

4. The Continuity Lemma

Good estimates of the distance between two points in \( T(S) \), of a compact Riemann surface of \( g \geq 1 \), will be crucial in our arguments. The following Lemma, due to Garsia ([17]), serves this purpose. In order to formulate it, we have to fix, in the holomorphic universal covering space \( \tilde{S} \) of \( S \), a fundamental domain \( P \) for the covering group \( G \). Assume that \( \omega \in Q_1(S) \) is a local coordinate for a neighborhood of \([id_S]\) in \( T(S) \) provided \( \| \omega \| \leq 2\epsilon < 1 \).

If \( f_\omega : S \to S_\omega \) is a quasiconformal map and \([f_\omega] \in T(S)\), then we write \([f_\omega] = \omega\). Let \( f_0 : S \to S_0 \) be a homeomorphism so that \([f_0] \in T(S)\). Assume that \([f_0] = \omega_0 \) and denote by \( B_\epsilon(\omega_0) \subset Q_1(S) \) the set of elements in \( T(S) \) with \( \| \omega - \omega_0 \| < \epsilon \).

Lemma 4.1. (Garsia [17]) If \([f_\omega] \in B_\epsilon(\omega_0) \) and if there is a quasiconformal mapping \( \chi : S_\omega \to S_{\omega'} \), whose dilatation \( K_\chi \) satisfies
\[
(1) \ K_\chi \leq K_0,
(2) \ K_\chi \leq 1 + \delta \quad \text{except on} \ A \subset P \quad \text{and}
(3) \ \text{area} \ A \leq \varsigma,
\]
then there is a constant \( b = b(K_0, \delta, \varsigma) \) so that
\[
\| \omega' - \omega \| \leq b(K_0, \delta, \varsigma).
\]
Further, if \( K_0 \) is bounded as \( (\delta, \varsigma) \to (0,0) \), then \( b(K_0, \delta, \varsigma) \to 0 \).
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Proof. See Garsia [17, p. 100 ff]. □

5. Guides to the Proofs of the Theorems 2.1 and 2.2

Let $\mathcal{M}$ be an orientable Riemannian manifold of dim $\mathcal{M} \geq 3$ and let $S$ be a $C^\infty$-embedded compact Riemann surface, of genus $g \geq 1$, in $\mathcal{M}$. Assume that $S_0$ is any fixed Riemann surface structure on $S$.

We begin with considerations of supporting lemmas.

5.1. Two supporting lemmas. First we examine the question of the existence of smooth nowhere vanishing sections of the normal bundle $NS$ for the Riemann surface $S$ embedded in the Riemannian manifold $\mathcal{M}$.

Lemma 5.1 (Existence of smooth nowhere vanishing section of $NS$). If $\mathcal{M}$ is an orientable Riemannian manifold and dim $\mathcal{M} = 3$ or $\geq 5$, then there always exists a nowhere vanishing smooth section of the normal bundle of $S$ in $\mathcal{M}$. If dim $\mathcal{M} = 4$, then there exists a nowhere vanishing smooth section of the normal bundle of $S$ in $\mathcal{M}$ if the Euler class of $NS$ vanishes.

Proof. In dimension 3, this is a well-known fact in differential topology (see Chen [10, p. 37] or O’Neill [33, p. 189]). If dim $\mathcal{M} \geq 5$, we have the following argument. The zero section and the desired section are both maps of the surface into the normal bundle. Since the codimension of the images is bigger than 2, we can assume that the intersections are all locally transverse, so the intersection of a section $\Gamma$ with the zero section consists of isolated points in the section. Form a little bubble in a normal direction at a zero. This forms a bubble over the zero section and we are done.

When dim $\mathcal{M} = 4$, the obstruction lies in the second cohomology class $H^2(S; \mathbb{Z})$ and it is the Euler class $e(\text{NS}) \in H^2(S; \mathbb{Z})$ of $\text{NS}$ of the surface $S$ in $\mathcal{M}$. If $e(\text{NS}) = 0$, then there exists such a nowhere vanishing section of the normal bundle $\text{NS}$. For example if $\mathcal{M} = \mathbb{R}^4$ or $\mathcal{M}$ has a trivial tangent bundle, then nonzero sections exist. (See Steenrod [39, Sec. 6, 12, 38, 39], Hirsch [19, p. 131ff] or Milnor and Stasheff [29].) □

By explicit construction we will show that the set of moduli corresponding to $\epsilon$-normal deformations of $S$ is dense in $T(S)$. To obtain that result, we will need the following Lemma.

Lemma 5.2. Suppose that a continuous mapping

$$f : \mathcal{B}_s(p_0) = \left\{ p \in \mathbb{B}^k : |p - p_0| \leq s \right\} \to \mathbb{B}^k$$

...
has the property
\[(5.1) \quad |f(p) - p| \leq s, \text{ for all } p \in \mathcal{B}_s(p_0)\]
for each \(p_0 \in \mathcal{B}^k\). Then
\[f(p_1) = p_0 \text{ for some } p_1 \in \mathcal{B}_s(p_0)\].

**Proof.** Let
\[g_1(p) = p + p_0\]
and
\[g_2(p) = p - f(p),\]
then
\[g = g_2 \circ g_1 : p \mapsto p + p_0 - f(p + p_0)\]
is a continuous mapping of \(\mathcal{B}_s(0)\) onto itself since for \(p \in \mathcal{B}_s(0), p + p_0 \in \mathcal{B}_s(p_0)\) and
\[(1) \quad p \mapsto f(p + p_0) = (f \circ g_1)(p) \text{ is continuous as a composite of continuous functions, and} \]
\[(2) \quad \text{by (5.1), } |p + p_0 - f(p + p_0)| \leq s, \text{ for each } p \in \mathcal{B}_s(0).\]
So the Brouwer’s fixed point theorem ([21, p. 40]) applied to the mapping
\[g : p \mapsto p_0 + p - f(p + p_0)\]
implies that there exists a point \(p' \in \mathcal{B}_s(0)\) such that
\[p' = p_0 + p' - f(p' + p_0), \text{ i.e., } f(p' + p_0) = p_0.\]
Therefore there is a point \(p_1 = p' + p_0 \in \mathcal{B}_s(p_0)\) such that \(f(p_1) = p_0\). \(\square\)

5.2. **A Guide to the Proof of the Main Theorem 2.1.** In Section 2.1, we defined an \(\epsilon\)-normal deformation \(S_h\) of \(S\) in an orientable Riemannian manifold \(\mathfrak{M}\). We follow the notation of Sections 2.1 and 4.

In Section 6, we introduce a process which is defined when \(\omega\) is restricted to a compact subset of \(Q_1(S) \setminus \{0\}\). It uses a family of the metrics given by \(dX^2 = \lambda^2|dz|^2\) (for \(X\) is a conformal parameter) to generate a family of smooth deformations in \(\mathfrak{M}\) of the surface \(S\).

**The existence of the family of deformations.** The first part of Theorem 2.1, the existence of \(\epsilon = \epsilon(S)\) for each Riemann surface \(S\) in \(\mathfrak{M}\) which guarantees the existence of a family of embedded \(\epsilon\)-normal deformations \(\{S_h\}\) of \(S\), is proved in Lemma 6.1. As we have seen, this only requires the existence of a nowhere vanishing section of the normal bundle of \(S\).
The existence of the conformal model. First we fix a map $h : S \times B_\epsilon(\omega_0) \to (-\epsilon, \epsilon)$ so that $h$ is a $C^\infty$-function on $S$ for each fixed $\omega$. Denote by $[\mathcal{S}^\omega] = \left[ (S^\omega, \bar{\mathcal{S}}^\omega) \right]$ the conformal equivalence class of the surface $S^\omega$ as a marked surface $(S^\omega, \bar{\mathcal{S}}^\omega)$ (see Definition 2 and foregoing discussions for details). We then define a map $\Xi$ of $B_\epsilon(\omega_0)$ to $T(S)$ by

$$\Xi : B_\epsilon(\omega_0) \to T(S) \quad \omega \mapsto [\mathcal{S}^\omega].$$

Here the surface $S^\omega$ is the $\epsilon$-normal deformation of $S$ defined by the map $S^\omega := S_h(\cdot, \omega) : S \to M \subset \mathbb{R}^m$

$$X(z) \mapsto \alpha_{X(z)}(1) = \beta(X(z), h(X(z))) = X(z) + h(X(z), \omega) \tilde{\Gamma}(X(z)) + r(h^2),$$

where the remainder term $r(h^2)$ is $O(h^2)$ as in equation (2.2). Then, as a consequence of Lemma 5.2, we will have proved the existence of the conformal model if we can prove that, given $[\mathcal{F}_0] = \omega_0$ and $\epsilon > 0$, for $\omega$ in the closed ball $B_\epsilon(\omega_0) \subset Q_1(S)$, there is a family of deformations $S^\omega$ of $S$ depending on parameters $\omega \in B_\epsilon(\omega_0)$ so that the following is true.

Lemma 5.3 (Dependence of $S^\omega$ on Parameters $\omega$). In the above notation,

1. $\Xi : \omega \mapsto [\mathcal{S}^\omega]$ is continuous in $B_\epsilon(\omega_0)$.
2. $\| [\mathcal{S}^\omega] - [id_\omega] \| \leq \epsilon, \forall \omega \in B_\epsilon(\omega_0)$, where $id_\omega : S \to S_\omega$ is the set-theoretic identity map.

Garsia’s Continuity Lemma (Lemma 4.1) implies that the family $\{S^\omega\}$ satisfies property (1) if the coefficients of $(d\mathcal{S}^\omega)^2$ depend continuously on $(z, \omega) \in \bar{S} \times B_\epsilon(\omega_0)$. We give an explicit formula for the functions $h(\cdot, \omega)$ in Section 6.3 and 6.4; from the formulas it follows directly that this property is satisfied (see Lemma 6.10).

To prove property (2), we let $\chi = \mathcal{S}^\omega \circ (id_\omega)^{-1} : S_\omega \to S^\omega$, then its dilatation $K_\chi$ satisfies (by the formula (3.4))

$$K_\chi^2 = \sup \frac{(d\mathcal{S}^\omega)^2}{(d\mathcal{S}^\omega)^2_{\omega}} \inf \frac{(d\mathcal{S}^\omega)^2}{(d\mathcal{S}^\omega)^2_{\omega}},$$

where both the supremum and infimum are taken over all directions and $d\mathcal{S}^\omega_{\omega}$ is as defined in (3.3). The computation of $K_\chi$ is given in Lemma 6.10. We construct the set $A$ in (6.12) and (6.40) [in Sections 6.3 and 6.4]. We compute the constant $\varsigma$, the maximum area of $A$, in (6.20) and (6.47) [in Sections 6.3 and 6.4]. We also
determine the constant $\delta$ in Lemma 6.10 [in Section 6.5] and the constant $\epsilon$ in Section 6.6, for which we get $b(K_0, \delta, \varsigma) \leq \epsilon$ in Garsia’s Continuity Lemma 4.1.

Then application of Garsia’s Continuity Lemma gives property (2).

Continuation of the outline of the proof of the existence of the conformal model. By Lemma 5.3, the function $\Xi$ satisfies the hypotheses of Lemma 5.2. Therefore there is a point $\omega_1 \in \overline{B}_\epsilon(\omega_0)$ so that

$$\Xi(\omega_1) = [S^{\omega_1}] = \omega_0 = [f_0],$$

i.e., for this $\omega_1 \in \overline{B}_\epsilon(\omega_0)$, the deformed surface $S^{\omega_1}$ can be mapped conformally onto $S_0$ by a mapping homotopic to $f_0 \circ (S^{\omega_1})^{-1}$.

Finally in Section 6.6, we collect all the facts needed to finish the proof of Theorem 2.1.

5.3. Proof of Theorem 2.2. Let $Maps_\epsilon(S)$ be the set given in the Section 2.1. Let $z$ be a conformal local coordinate on $S$. In this local coordinate, we define metrics $\rho$ on $Maps_\epsilon(S)$. For $f$ and $g \in Maps_\epsilon(S)$:

1. The $C^1$-topology is defined by:

$$\rho_1(f, g) = \|f - g\|_\infty + \left\| \frac{\partial}{\partial z} (f - g) \right\|_\infty + \left\| \frac{\partial}{\partial \bar{z}} (f - g) \right\|_\infty.$$

2. The $C^0$-topology is defined by:

$$\rho_0(f, g) = \|f - g\|_\infty.$$

Proof (of Theorem 2.2) Define the canonical map $\Upsilon$ of $Maps_\epsilon(S)$ onto $T(S)$ by

$$\Upsilon : Maps_\epsilon(S) \longrightarrow T(S)$$

$$f \mapsto [f].$$

The continuity of $\Upsilon$ is equivalent to the convergence of the sequence $\{[f_n]\}$ to $[f]$ for every convergent sequence $f_n \rightarrow f$ in $Maps_\epsilon(S)$.

1. In the $C^1$-topology: Let $\{f_n\}$ be a sequence in $Maps_\epsilon(S)$ converging to $f \in Maps_\epsilon(S)$, then

$$\rho_1(f_n, f) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

That is

$$f_n \rightarrow f \quad \text{as} \quad n \rightarrow \infty.$$
and
\[ \frac{\partial f_n}{\partial z} \to \frac{\partial f}{\partial z} \quad \text{and} \quad \frac{\partial f_n}{\partial \bar{z}} \to \frac{\partial f}{\partial \bar{z}} \quad \text{uniformly as} \quad n \to \infty. \]

Hence, for complex dilatation \( \mu_{f_n} = \frac{(f_n)_z}{(f_n)_{\bar{z}}} \), we obtain
\begin{equation}
(5.8) \quad \mu_{f_n} \to \mu_f \quad \text{uniformly as} \quad n \to \infty.
\end{equation}

On the other hand, the Teichmüller distance between \([f_n]\) and \([f]\) is given by
\begin{equation}
(5.9) \quad d([f_n], [f]) = \inf \left\{ \frac{1}{2} \log \sup_h K_{h_n}(z), \quad h_n \simeq f \circ f_n^{-1} \right\},
\end{equation}
where \(K_{h_n}\) is the dilatation of \(h_n\) and \(\simeq\) denotes free homotopy.

For complex dilatations \(\mu_{f \circ f_n^{-1}}\) and \(\mu_{f^{-1}}\), we have
\begin{equation}
(5.10) \quad \mu_{f \circ f_n^{-1}} = \frac{\mu_{f^{-1}} + (\mu_f \circ f_n^{-1}) \cdot \left( \frac{(f_n^{-1})_z}{(f_n^{-1})_{\bar{z}}} \right)}{1 + \mu_{f^{-1}} \cdot (\mu_f \circ f_n^{-1}) \cdot \left( \frac{(f_n^{-1})_z}{(f_n^{-1})_{\bar{z}}} \right)},
\end{equation}
and
\begin{equation}
(5.11) \quad (\mu_f \circ f^{-1}) \frac{(f^{-1})_z}{(f^{-1})_{\bar{z}}} = -\mu_{f^{-1}}.
\end{equation}

(For both computations, see Ahlfors [3, p. 8-9].)

Since \(\mu_{f_n} \to \mu_f\) and \(f_n \to f\) as \(n \to \infty\), from equations (5.7), (5.10) and (5.11), we obtain
\[ \mu_{f \circ f_n^{-1}} \to 0 \quad \text{as} \quad n \to \infty \]
and consequently from formula (5.9)
\[ d([f_n], [f]) \to 0 \quad \text{as} \quad n \to \infty. \]

Therefore \([\{f_n\}] \to [f]\) in \(T(S)\).

2. In the \(C^0\)-topology: In this case, we do not necessarily have bounds on the size of derivative so we cannot control the size of the dilatation \(K_{h_n}(z)\). Therefore we may not have a convergent sequence \([f_n]\) in \(T(S)\) even if the sequence \(\{f_n\}\) converges in \(Maps(S)\). More precisely, choose a sequence \(\epsilon_n \to 0\). The Main Theorem (Theorem 2.1) tells us that for each \(n\),
\[ \Upsilon(B_{\epsilon_n}(id)) = T(S). \]

So there exists a sequence \(\{f_n\} \in Maps(S)\) so that \(\rho_0(f_n, id) < \epsilon_n\) but the Teichmüller distance between \([f_n]\) and \([id]\) is bigger than 1. Thus \(\Upsilon\) is not continuous in the \(C^0\)-topology. \(\square\)
6. The Construction of the family $S^\omega$

We will assume that the genus $g$ of a compact Riemann surface $S$ always is $\geq 1$ since genus 0 compact Riemann surfaces are all conformally equivalent. Let $\tilde{S}$ be the holomorphic universal covering of $S$. Then we may assume $\tilde{S} = \mathbb{C}$ if $g = 1$ or $\tilde{S} = \Delta$ if $g > 1$. Also let $G$ be the group of deck transformations and $P$ be a fundamental domain for $G$. We may assume that $\partial P$ has measure zero and the fundamental domain $P$ is compact in $\tilde{S}$ since $S$ is compact (see Lehner [27, p. 203-205]).

In Section 3, we identified the Teichmüller space $T(S)$ with $Q_1(S)$. Recall that $X : \tilde{S} \to \mathcal{M} \subset \mathbb{R}^m$ is a conformal parametrization of $S$ in $\mathcal{M} \subset \mathbb{R}^m$. This permits us to identify the point $X(z) \in S$ with the point $z \in P$. We will not have to worry about the parametrization being two-to-one along $\partial P$. We follow the notation of Section 5.2.

In several places we will define two different functions, one in genus $g = 1$ and one in genus $g > 1$. We will use the same notation for the two functions in order to obtain a treatment valid in all genera.

6.1. The Metric $d\sigma_\omega^2$.

For each nonzero holomorphic quadratic differential $\omega = \phi_\omega(z)dz^2 \in Q_1(S) \setminus \{0\}$, define a metric

$$d\sigma_\omega^2 = \lambda^2 \cdot |dz + \Psi_\omega(z)d\bar{z}|^2,$$

where

$$\Psi_\omega(z) = \|\omega\| \frac{\phi_\omega(z)}{|\phi_\omega(z)|}$$

and $\lambda^2 > 0$ is a smooth real-valued $(1, 1)$–form.

As in Section 3, the above metric (6.1) is well-defined. Note that $\|\Psi_\omega\|_\infty = \|\omega\|$. Since

$$|dz + \Psi_\omega(z)d\bar{z}| \leq |dz| + |\Psi_\omega(z)||d\bar{z}| \leq (1 + \|\Psi_\omega\|_\infty)|dz|,$$

the metric (6.1) satisfies the following inequality:

$$\lambda^2(z)|dz + \Psi_\omega(z)d\bar{z}|^2 \leq \lambda^2(z)(1 + \|\Psi_\omega\|_\infty)^2|dz|^2.$$

The metric (6.1) defines a new conformal structure on $S$ which will be denoted by $S_\omega = (S, \omega/\|\omega\|, \|\omega\|)$. The metric of the $\epsilon$–normal deformation $S_h$, defined by the map $S_h$ in equation (2.2), of $S$ satisfies the equation

$$(dS_h)^2 = \lambda^2(z)|dz|^2 + (dh)^2 + o(h)|dz|^2.$$
Here we use the uniformizing variable $z \in \tilde{S}$ for $(dX)^2 = \lambda^2|dz|^2$. Let $\chi : S_\omega \to S_h$ be a mapping of $S_\omega$ onto $S_h$. Let $ds^2_\omega$, given by (6.1), and $(dS_h)^2$, given in (6.4), be metrics for the surface $S_\omega$ and the surface $S_h$ respectively. We want to show that the dilatation $K_\chi$ of the map $\chi$ satisfies the hypotheses of Garsia’s Continuity Lemma 4.1. We derived an expression for $K_\chi$ in equation (5.2). It will be helpful to split $ds^2_\omega$ into the form given in (6.4).

Let

\begin{equation}
\Pi(\omega) = \{ z \in \tilde{S}, \Im \phi_\omega(z) \neq 0 \}
\end{equation}

if $g > 1$ or $\Pi(\omega) = C$ if $g = 1$, where $\Im \phi_\omega(z)$ is an imaginary part of nonzero holomorphic function $\phi_\omega(z)$. The metric $ds^2_\omega$ is smooth on the set $\Pi(\omega)$. Let

\begin{equation}
\gamma_\omega := (1 - \|\Psi_\omega\|)_\infty^{-2} = (1 - \|\omega\|)_\infty^{-2}.
\end{equation}

Then define the following real valued functions on $\Pi(\omega)$.

\begin{align}
\alpha^2_\omega & := 2\gamma_\omega (\|\Psi_\omega\|_\infty + \Re \Psi_\omega(z)) = 2\gamma_\omega (\|\omega\| + \Re \Psi_\omega(z)), \\
\beta^2_\omega & := 2\gamma_\omega (\|\Psi_\omega\|_\infty - \Re \Psi_\omega(z)) = 2\gamma_\omega (\|\omega\| - \Re \Psi_\omega(z)),
\end{align}

where $\Re \Psi_\omega(z)$ is a real part of nonzero function $\Psi_\omega(z)$.

Notice that if $S$ has genus 1, then nonzero holomorphic function $\phi_\omega(z)$ is constant (see the Theorem 3.4 and thereafter), hence $\Psi_\omega(z)$, $\alpha_\omega(z)$ and $\beta_\omega(z)$ are constants.

On each connected component of $\Pi(\omega)$, choose continuous real branches of $\alpha_\omega, \beta_\omega$, so that

\begin{equation}
\sgn (\alpha_\omega \beta_\omega) = \sgn (\Im \Psi_\omega(z)) \quad \text{and} \quad \beta_\omega > 0.
\end{equation}

Since $dx^2 = dx^2 - dy^2 + 2i dx dy$ and $dz^2 = dx^2 - dy^2 - 2i dx dy$, we get

\begin{equation}
\gamma_\omega ds^2_\omega = \lambda^2(z) (|dz|^2 + (\alpha_\omega dx + \beta_\omega dy)^2) .
\end{equation}

6.2. The Stability of the Deformed Surface. We now prove the ‘Existence of normal deformations of $S$’ part of Theorem 2.1. We follow the notation of Section 2.1.

**Lemma 6.1.** Suppose $S$ is a Riemann surface in $\mathcal{M}$. Assume that there exists a smooth nowhere vanishing section $\Gamma$ of the normal bundle $NS$ of $S$ in $\mathcal{M}$. Then there is an $\epsilon_0 > 0$ such that, for all $C^\infty$-functions $h$ defined on $S$ with

$$
\|h\|_\infty < \epsilon_0,
$$

there exists an $\epsilon_0$-normal deformation $S_h$ of $S$ and $S_h$ is a $C^\infty$-embedded surface.
Proof. By assumption, the maps $X(z)$, $h$ and
\[ \beta(a, b(a)) = a + b(a)\overline{\Gamma(a)} + r(b^2(a)) \]
are differentiable, so
\[ \mathcal{S}_{h}(z) = \beta(X(z), h(X(z))) = X(z) + h(X(z))\overline{\Gamma(X(z))} + r(h^2(X(z))) \]
is differentiable.

By the property of the exponential map, there exists an $\epsilon_0 > 0$ so that, for $|r| < 2\epsilon_0$, the mapping
\[ \beta : \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \]
\[ (X(x, y), r) \longmapsto \exp r\Gamma(X(x, y)) \]
exists and is injective.

We then observe that the mapping
\[ (x, y, h) \longmapsto \mathcal{S}_{h}(x, y) = \beta(X(x, y), h(X(x, y))) \]
exists and is injective. That is, the surfaces $S_h$ with $\|h\|_\infty < \epsilon_0$ are $C^\infty$-embedded $\epsilon_0$-normal deformations of $S$.

\[ \square \]

6.3. The Deformation Function $h$ for $g = 1$. According to the previous sections, to complete the Embedding Theorem 2.1, we need to describe a deformation function $h : S \rightarrow (-\epsilon, \epsilon)$ which satisfies the following properties:

1. $h$ is $C^\infty$.
2. $\|h\|_\infty < \epsilon$, for example take $\epsilon < \epsilon_0$, where $\epsilon_0$ is given by the previous Lemma.
3. $(dh)^2$ is proportional to $(\alpha_\omega dx + \beta_\omega dy)^2$ in view of equations (6.4) and (6.9).

We would like to define a function $h$ which satisfies condition (3) except on a sufficiently small set. We are tempted to write $h = \alpha_\omega x + \beta_\omega y$, but this function will usually violate the condition (2). Therefore we must abandon global linearity and get a smooth approximation to a piecewise linear function $h$. We wish to apply Garsia’s Continuity Lemma 4.1. The hypotheses of the Lemma would be simple if we could apply the saw-tooth function with slope $\pm 1$ to $\alpha_\omega x + \beta_\omega y$. The sign change doesn’t affect condition (3) except at the corners and this can be smoothed away on a small set. On the other hand, near the tip, condition (1) is violated. A smoothing procedure gives the necessary improvements. But then condition (3) is again broken. Again we smooth away the problem on a small set. For $h$ to be well-defined on $S$, it is convenient that it be zero in a neighborhood of
the edges of $P$ but remain smooth. (In this $g = 1$ case, without loss of generality, we take $P = [0, 1]^2$.)

We now proceed to define the smoothing procedure. In Section 6.5, we will compute the necessary estimates. Suppose $\eta$ is a fixed small number less than $\frac{1}{16}$. We shall define two auxiliary real-valued differentiable functions as follows:

a. $\mu_\eta(x)$:
   \begin{enumerate}
   \item $0 \leq \mu_\eta \leq 1$,
   \item $\mu_\eta(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq \frac{\eta}{2} \\ 1 & \text{for } \frac{\eta}{2} \leq x \leq \frac{1}{2}, \end{cases}$
   \item $\mu_\eta(1 - x) = \mu_\eta(x)$.
   \end{enumerate}

b. $\nu_\eta(x)$:
   \begin{enumerate}
   \item $|\dot{\nu}_\eta(x)| \leq 1$, (derivative $\dot{\nu}_\eta(x)$ is taken with respect to the variables in it.)
   \item $\nu_\eta(x) = \begin{cases} x & \text{for } \eta - 1 \leq x \leq 1 - \eta \\ 2 - x & \text{for } 1 + \eta \leq x \leq 3 - \eta, \end{cases}$
   \item $\nu_\eta(x + 4) = \nu_\eta(x)$.
   \end{enumerate}

Let $F$ be a compact set in $Q_1(S)$ which does not contain 0. Let $N_F$ be an integer to be determined later in this section and $\epsilon$ be the constant given in the Theorem 2.1. For $(x, y) \in P$, $\omega \in F$ and

$$N > N_F + \frac{1}{\epsilon} \cdot \max_{z \in P} |\lambda(z)|$$

with the constant $\epsilon$ of Theorem 2.1, we define

\begin{equation}
(6.10) \quad h(x, y, \omega, N) = \frac{1}{N} \mu_\eta(x) \mu_\eta(y) \lambda(x, y) \nu_\eta(N \cdot (\alpha \omega x + \beta \omega y))
\end{equation}

Then $h$ is $C^\infty$ on $P$ and continuous on $P \times F$. We get

\begin{equation}
(6.11) \quad (dh)^2 = \lambda^2 \mu_\eta^2(x) \mu_\eta^2(y) \nu_\eta^2(N \cdot (\alpha \omega x + \beta \omega y))(\alpha \omega dx + \beta \omega dy)^2 + o\left(\frac{1}{N}\right)|dz|^2,
\end{equation}

where $o\left(\frac{1}{N}\right) \to 0$ uniformly on $P \times F$ as $N \to \infty$. This follows because all the previous functions are continuous on this compact set and $\frac{1}{N}$ multiplies them all.

In the Section 6.5, we will check the conditions of the Garsia’s Continuity lemma 4.1. So we need to show the area of exceptional set

\begin{equation}
(6.12) \quad A = \{ (x, y) \in [0, 1]^2 \mid \mu_\eta(x) \cdot \mu_\eta(y) \cdot \nu_\eta^2(N \cdot (\alpha \omega x + \beta \omega y)) \neq 1 \}
\end{equation}

can be made arbitrarily small.
Let
\[(6.13) \quad A_\mu = \{(x, y) \in [0, 1]^2 \mid \mu_\eta(x) \cdot \mu_\eta(y) \neq 1\},\]
then its Euclidean area \(\text{vol } A_\mu\) satisfies
\[\text{vol } A_\mu < 4\eta(1 - \eta).\]

The proof of this last statement is a trivial observation.

We next show

**Lemma 6.2.** If
\[(6.14) \quad A_N = \{(x, y) \in [0, 1]^2 \mid \nabla^2 (N \cdot (\alpha_\omega x + \beta_\omega y)) \neq 1\},\]
then for \(N > 2\sqrt{2}/\sqrt{\|\omega\|}\), the Euclidean area \(\text{vol } A_N\) satisfies
\[(6.15) \quad \text{vol } A_N < 2\eta.\]

**Proof.** Let \(\text{vol } (\cdot)\) denote the Euclidean area of a set and, for brevity, let \(\alpha = \alpha_\omega\) and \(\beta = \beta_\omega\).

We have
\[\nabla^2 (a) \neq 1 \text{ only if } |a - (2m + 1)| \leq \eta.\]

For \(a = N \cdot (\alpha x + \beta y)\), this condition is equivalent to the inequalities
\[1 - \eta + 2m \leq N \cdot (\alpha x + \beta y) \leq 1 + \eta + 2m, \quad \text{for } m \in \mathbb{Z}.\]

These specify the set \(A_N\), which is a collection of equidistant strips each defined by the previous inequalities for some fixed \(m\). Denote by \(\sigma_0\) (respectively \(\sigma_n\)) the first strip (respectively the last strip) which intersect \(P\) in an open non-empty set.

(1) We first compute the ratio \(W/C\), where \(W\) is the width of a strip \(\sigma_m\), \(m = 0, \cdots n\) and \(C\) is the distance between center lines of two neighboring strips.
\[\frac{W}{C} = \frac{2\eta}{N \cdot \sqrt{\alpha^2 + \beta^2}} \cdot \frac{2}{N \cdot \sqrt{\alpha^2 + \beta^2}} = \eta/1 = \eta.\]

(2) Let \(R_m\) be the region between center lines of two neighboring strips \(\sigma_m\) and \(\sigma_{m+1}\). Assume that \(R_{-1}\) is the triangle below the region \(R_0\) and \(R_n\) is the triangle above the region \(R_{n-1}\).

Let \(l_m\) (respectively \(u_m\)) be the length of lower side (respectively upper side) of \(\sigma_m\) and let \(l'_m\) (respectively \(u'_m\)) be the length of lower side (respectively upper side) of \(R_m\). Denote by \(A_m = \text{vol } (\sigma_m)\) the area of \(\sigma_m\) and by \(D_m = \text{vol } (R_m)\) the area of \(R_m\).
We first pick numbers $m_0$ and $m_1$ so that

$$(0, 1) \in \overline{\sigma_{m_0} \cup R_{m_0}} \text{ and } (1, 0) \in \overline{\sigma_{m_1} \cup R_{m_1}}.$$ 

By symmetry, we may assume that $m_0 \leq m_1$.

Then for sufficiently large $N$, we have

$$l_m < l_m' \text{ and } u_m < u_m' \text{ if } m < m_0,$$

$$l_m = l_m' \text{ and } u_m = u_m' \text{ if } m_0 < m < m_1,$$

$$l_{m+1} < l_m' \text{ and } u_{m+1} < u_m' \text{ if } m > m_1.$$ 

By (1), we know that $W = \eta C$. The area of a parallelogram whose base and top are parallel and have lengths $b$ and $t$ respectively is given as

$$\frac{1}{2} h(b + t),$$

where $h$ is the height of the parallelogram. For the area of all strips except $\sigma_{m_0} \cup \sigma_{m_1}$, we then obtain

$$\sum_{0}^{n} A_m := \sum_{m=0}^{m_0-1} A_m + \sum_{m=m_0+1}^{m_1-1} A_m + \sum_{m=m_1+2}^{n} A_m + A_{m_1+1}$$

$$= \frac{\eta C}{2} (l_{m_{1}+1} + u_{m_{1}+1}) + \sum_{m=0}^{m_{0}-1} \eta C \cdot \frac{1}{2} (l_m + u_m)$$

$$+ \sum_{m=m_{0}+1}^{m_{1}-1} \eta C \cdot \frac{1}{2} (l_m + u_m) + \sum_{m=m_{1}+2}^{n} \eta C \cdot \frac{1}{2} (l_m + u_m)$$

$$= \frac{1}{2} \eta C \left( l_{m_{1}+1} + u_{m_{1}+1} \right) + \sum_{m=m_{0}+1}^{m_{1}-1} (l_m + u_m)$$

$$+ \sum_{m=m_{0}+1}^{m_{1}-1} (l_m + u_m) + \sum_{m=m_{1}+1}^{m_{1}+2} (l_{m_{1}+1} + u_{m_{1}+1}) \right).$$

(We use $\sum'$ to denote a sum over $m \in \{0, \ldots, n\} \setminus \{m_0, m_1\}$.)
Also for the area \( \text{vol}\ (\cup R_m) \) of all \( R_m \), we get

\[
\sum_{m=1}^{n} D_m \quad = \quad D_{-1} + D_{m_0} + D_{m_1} + D_{n} + \sum_{m=0}^{m_0-1} D_m + \sum_{m=m_0+1}^{m_1-1} D_m + \sum_{m=m_1+1}^{n-1} D_m \\
\quad = \quad D_{-1} + D_{m_0} + D_{m_1} + D_{n} + \sum_{m=0}^{m_1-1} C \cdot \frac{1}{2} (l_m' + u_m') \\
+ \sum_{m=m_0+1}^{m_1-1} C \cdot \frac{1}{2} (l_m' + u_m') + \sum_{m=m_1+1}^{n-1} C \cdot \frac{1}{2} (l_m' + u_m')
\]

(6.18)

\[
= \frac{C}{2} \left( l + \sum_{m=0}^{m_0-1} (l_m' + u_m') + \sum_{m=m_0+1}^{m_1-1} (l_m' + u_m') \\
+ \sum_{m=m_1+1}^{n-1} (l_m' + u_m') \right),
\]

where \( l \) is some number which satisfies

\[
\frac{1}{2} l C = D_{-1} + D_{m_0} + D_{m_1} + D_{n}.
\]

Therefore from equations (6.17) and (6.18), we have

\[
\frac{\text{vol} (A_N \setminus \{\sigma_{m_0} \cup \sigma_{m_1}\})}{\text{vol} (\cup R_m)} = \frac{\sum_{m=0}^{n} A_m}{\sum_{m=-1}^{n} D_m} \leq \eta \frac{L}{R},
\]

(6.19)

where

\[
L \quad = \quad (l_{m_1+1} + u_{m_1+1}) + \sum_{m=0}^{m_0-1} (l_m + u_m) + \sum_{m=m_0+1}^{m_1-1} (l_m + u_m) \\
+ \sum_{m=m_1+1}^{n-1} (l_m + u_m),
\]

\[
R \quad = \quad l + \sum_{m=0}^{m_0-1} (l_m' + u_m') + \sum_{m=m_0+1}^{m_1-1} (l_m' + u_m') + \sum_{m=m_1+1}^{n-1} (l_m' + u_m').
\]

To get the relationship between \( l \) and \( l_{m_1+1} + u_{m_1+1} \), we note the following:

\( l_m' + l_m' > l_{m_1+1} \) and \( u_m' + u_m' > u_{m_1+1} \)
so that
\[
\frac{1}{2} l C = D - D_m + D_{m_1} + D_n
\]
\[
\geq \frac{1}{2} C (l_m' + u_m' + u_{m_1}' + u_m'),
\]
therefore
\[
l \geq l_m' + u_m' + u_{m_1}' + u_m' > l_{m_1} + u_{m_1}.
\]
From this relation and equation (6.16), we get
\[
\frac{L}{R} < 1.
\]
Therefore from equation (6.19), we obtain
\[
\sum_{m=0}^{n} A_m = \sum_{m=0}^{n} \frac{A_m}{D_m} < \eta.
\]
Therefore
\[
\text{vol} (A_N) = \sum_{m=0}^{n} A_m = \sum_{m=0}^{n} A_m + \text{vol} (\sigma_{m_0}) + \text{vol} (\sigma_{m_1})
\]
\[
\leq \eta + \text{vol} (\sigma_{m_0}) + \text{vol} (\sigma_{m_1}).
\]
Consequently, vol (A_N) can differ from the area of strips \(\sigma_{m_0} \cup \sigma_{m_1}\) by at most \(\eta\).

For the area of the strip \(\sigma_{m_0}\), we obtain
\[
\text{vol} (\sigma_{m_0}) < \sqrt{2} \cdot \frac{2\eta}{N \cdot \sqrt{\alpha^2 + \beta^2}} \leq \frac{2\sqrt{2}\eta}{N \cdot 2\sqrt{\|\omega\|}} = \frac{\sqrt{2} \cdot \eta}{N \|\omega\|} < \frac{\eta}{2},
\]
where
\[
N > \frac{2\sqrt{2}}{\sqrt{\|\omega\|}}
\]
since
\[
\alpha^2 + \beta^2 = 4\gamma_\omega \|\omega\|.\]
This implies
\[
\sqrt{\alpha^2 + \beta^2} = \frac{2\sqrt{\|\omega\|}}{(1 - \|\omega\|)^2} \geq 2\sqrt{\|\omega\|}.
\]
The same computation gives the same bound \(\eta/2\) for vol (\(\sigma_{m_1}\)) of the strip \(\sigma_{m_1}\).

Finally we conclude that \(\text{vol} (A_N) < \eta + \eta/2 + \eta/2 = 2\eta.\) \(\square\)
In the proof, we restrict $\omega$ to a compact set $F$ in $Q_1(S)$ which does not contain $0$.

Therefore we can replace the condition $N > 2\sqrt{2}/\sqrt{\|\omega\|}$ by $N > N_F > 2\sqrt{2}/\sqrt{\|\omega\|}$. Then it is true, for $A = A_\mu \cup A_N$, where $A_\mu$ was defined in equation (6.13), that

\[(6.20) \quad \text{area} A \leq (\text{area} A_\mu) + (\text{area} A_N) < 2\eta(3 - 2\eta) \text{ for all } N > N_F.\]

Furthermore, with this bound of $N_F$, the inequality (6.20) is valid for $N > N_F + \frac{1}{2} \cdot \max_{z \in P} |\lambda(z)|$.

6.4. The Deformation Function $h$ for $g > 1$. At the beginning of Section 6.3, we listed the conditions that we want $h$ to satisfy. Condition (3) suggests that we express $(dh)^2$ in terms of constants $\alpha_\omega$ and $\beta_\omega$. In contrast to the case $g = 1$, in higher genera $\alpha_\omega$ and $\beta_\omega$ must be non-constant functions of $z$. The definition of $h$, on the other hand, will come as a solution of a differential equation in which $\alpha_\omega$, $\beta_\omega$ and their derivatives appear as coefficients. In order to get a $C^\infty$ solution, we need $\alpha_\omega$, $\beta_\omega$ to be smooth on all of $P$. Also they, together with their derivatives, must change as little as possible.

In this section, we will eventually construct the deformation function $h$ for $g > 1$ in terms of $\lambda(z), \alpha_\omega(z), \beta_\omega(z)$ and some large number $N$. But better yet, we need to do some preparatory works.

First of all, the functions $\alpha_\omega, \beta_\omega$ are not yet defined on $N_\omega = \{z \in \Delta, 3\phi_\omega(z) = 0\}$, where $\phi_\omega(z)$ is nonzero holomorphic function of $z$ used to define the metric $ds_\omega^2$ in (6.1).

To define them on $N_\omega$, we need several lemmas.

6.4.1. Preparatory lemmas. We consider $\phi_\omega(z)$ as a function of $(z, \omega) \in \Delta \times Q_1(S)$, therefore we denote it by $\phi(z, \omega)$ and list its properties here:

(a) $\phi(z, \omega)$ and $\phi'(z, \omega) := \partial\phi(z, \omega)/\partial z$ are real-analytic on $\Delta \times (Q_1(S) \setminus \{0\})$,

(b) For each fixed $\omega \in Q_1(S) \setminus \{0\}, \phi(z, \omega)$ is holomorphic and non-constant in $\Delta$.

Lemma 6.3. Let $K \subset \Delta$ be compact and $\omega \in Q_1(S) \setminus \{0\}$. Then the set

$$\Gamma(a, \omega) := \{z \in K | \phi(z, \omega) = a\},$$

varies continuously on the interior of $K$ with $\omega$. 

Proof. The set $\Gamma(a, \omega)$ for fixed $a$ and $\omega$ is discrete in $\Delta$. For fixed choice of global coordinate $z$ in $\Delta$, $\omega$ uniquely defines $\phi_\omega(z)$. Let $\Phi : Q_1(S) \to O(\Delta)$ be a mapping defined by $\omega \mapsto \phi_\omega(z)$, then $\Phi$ is a linear isomorphism. Therefore $D\Phi$ is an isomorphism and this implies that $\det D\Phi \neq 0$. By the Implicit Function Theorem, each zero of the function $\phi(z, \omega) - a$ varies continuously with $\omega$. Since $K$ is compact, there are only finitely many zeroes.

Lemma 6.4. Let $K \subset \Delta$ be compact and $\omega \in Q_1(S) \setminus \{0\}$, then the set

$$N_\omega \cap K := \{ z \in K \mid \Im \phi_\omega(z) = 0 \}$$

with Hausdorff topology (which is given by the metric $d(A, B) = \inf_{x \in A, y \in B} d(x, y)$, where $A$ and $B$ are non-empty subsets of $\Delta$) varies continuously on the interior of $K$ with $\omega$.

Proof. The same argument as in the previous Lemma works with $\Im \phi_\omega(z)$. In fact, this is a real-analytic version of the previous Lemma.

Lemma 6.5. Let $K \subset \Delta$ and $F \subset Q_1(S)$ be compact. Then for every $\eta > 0$, there exists a $\delta > 0$ such that if

$$N_\omega(\delta) := \{ z \mid \text{distance } (z, N_\omega) < \delta \}$$

is the tubular neighborhood of $N_\omega$ of width $\delta$, then the area of $K \cap N_\omega(\delta)$ is $< \eta$, for each $\omega \in F$.

Proof. As a neighborhood of a finite number of analytic curves, $K \cap N_\omega(\delta)$ has area which tends to zero with $\delta$. By outer measurability, the required $\delta$ exists.

Lemma 6.6. Suppose $K \subset \Delta$ and $F \subset Q_1(S)$ are compact. Then for a given $\delta > 0$, there exists a finite number of real numbers $x_1, \cdots, x_n$ such that if

$$U(x_i, \omega, \delta) = \bigcup_{z_{ij} \in \phi_\omega^{-1}(x_i) \cap K} \{ z \mid |z - z_{ij}| < \delta \},$$

then

$$K \cap N_\omega = \{ z \in K \mid \Im \phi_\omega(z) = 0 \} \subset \cup_{i=1}^n U(x_i, \omega, \delta), \text{ for all } \omega \in F.$$
6.4.2. Auxiliary functions. Recall that $\eta$ is a fixed small number less than $\frac{1}{16}$ and $P$ is a compact fundamental domain in $S$ for the group of deck transformations of the universal covering $\pi : \tilde{S} \to S$. Define a non-negative $C^\infty$ function $\mu(z)$ such that

$$
\mu(z) = \begin{cases} 
0 & \text{on a neighborhood } U_1 \text{ of } \Delta - P \\
1 & \text{outside a neighborhood } U \supset U_1,
\end{cases}
$$

where

$$
(6.21) \quad \text{area } (P \cap U) < \eta/4.
$$

And let $\tau(x)$ be a $C^\infty$-function for $x \geq 0$ so that $0 \leq \tau(x) \leq 1$ and

$$
\tau(x) = \begin{cases} 
0 & \text{for } x \leq 1 \\
1 & \text{for } x \geq 4.
\end{cases}
$$

By Lemma 6.5, for any compact set $F \subset Q_1(S)$, we can choose a $\delta' > 0$ such that

$$
(6.22) \quad \text{area } (N_\omega(\delta') \cap P) < \eta/4, \quad \text{for each } \omega \in F.
$$

Let $\delta = \frac{1}{3} \min \{\delta', \text{distance} (\partial P, \Delta - U_1)\}$.

By Lemma 6.6, there exist suitable real numbers $x_1, \cdots, x_n$ such that

$$
(6.23) \quad \bigcup_{i=1}^n U(x_i, \omega, \delta) \supset P \cap N_\omega, \quad \text{for all } \omega \in F.
$$

For $\omega \in F$ fixed,

$$
\Gamma_\omega = \{ z_{ij} \in P \mid \phi_\omega(z_{ij}) \in \bigcup_{i=1}^n \{ x_i \} \}.
$$

Then set

$$
\mu_\eta(z, \omega) := \mu(z) \prod_{z_{ij} \in \Gamma_\omega} \tau \left( \frac{|z - z_{ij}|^2}{\delta^2} \right).
$$

Then we have the following:

**Properties of the function $\mu_\eta(z, \omega)$ ;**

(1) \ $\mu_\eta(z, \omega) \equiv 0$ on \ $\bigcup_{i=1}^n U(x_i, \omega, \delta) \cup (\Delta \setminus P)$.

(2) \ $\mu_\eta(z, \omega) \equiv 1$ on $P - B$, where $B = (N_\omega(\delta') \cap P) \cup (P \cap U)$.

(3) \ area $B < \eta/2$.

(4) \ $\mu_\eta(z, \omega)$ is continuous on $\Delta \times F$.

(5) \ $\mu_\eta(z, \omega) = 0$ for $z \in N_\omega$.

**Proof.** (1) If $z \in \bigcup_{i=1}^n U(x_i, \omega, \delta)$, then $|z - z_{ij}| < \delta$ for some $z_{ij}$ by Lemma 6.6.

It follows that $\tau \left( \frac{|z - z_{ij}|^2}{\delta^2} \right) = 0$ by the definition of the function $\tau$ and the fact that $\mu_\eta(z, \omega) = 0$. 
(2) Since \( \delta < \frac{1}{2} \delta' \), we have \( \{ z \mid |z - z_{ij}| \delta^{-1} < 2 \} \subset N_{\omega}(\delta') \) for each \( i, j \). So, for every \( z \in P - B \), the inequality \( |z - z_{ij}|^2 \delta^{-2} \geq 4 \) is valid. We conclude that \( \mu_\eta(z, \omega) = 1 \) for all \( z \in P - B \) since \( \mu(z) = 1 \) in \( P - B \).

(3) From equations (6.21) and (6.22), we immediately obtain area \( B = \text{area} (\cup_{n=1}^{\infty} \{ 0 < |z - z_{ij}|^2 \delta^{-2} \leq 4 \}) < \eta/4 + \eta/4 = \eta/2 \).

(4) By Lemma 6.3, each factor of \( \mu_\eta \) is continuous on \( \Delta \times F \), so we only have to show it is a finite product. But the product is taken over the lifts to the compact set \( P \) of a finite number of points on \( S \). So the product is finite.

(5) Since we have \( \cup_{n=1}^{\infty} \cup_{i,j}^{\infty} U(x_i, \omega, \delta) \supset P \cap N_{\omega}, \Delta \setminus P \cap N_{\omega} \cap (\Delta \setminus P) \) and \( N_{\omega} = ((\Delta \setminus P) \cap N_{\omega}) \cup (P \cap N_{\omega}) \), this property follows from property (1).

\[ \square \]

Analogously, we can construct a \( C^\infty \) function \( \mu_\eta' \omega \) in \( \Delta \) such that

\[ \mu_\eta'(z, \omega) = \begin{cases} 0 & \text{in neighborhood of } (\Delta - P) \cup N_{\omega} \\ 1 & \text{if } \mu_\eta(z, \omega) \neq 0 \end{cases} \]

6.4.3. Construction of \( h \). For \( \alpha_\omega \) and \( \beta_\omega \) as in equation (6.7), let

\[ \tilde{\alpha}_\omega = \mu_\eta' \cdot \alpha_\omega \] and \[ \tilde{\beta}_\omega = \mu_\eta' \cdot \beta_\omega + 1 - \mu_\eta' \] for \( z \in \Delta \setminus N_{\omega} \)

and \[ \tilde{\alpha}_\omega = 0 \] and \[ \tilde{\beta}_\omega = 1 \] for \( z \in N_{\omega} \).

Since \( \mu_\eta', \alpha_\omega \) and \( \beta_\omega \) are \( C^\infty \) in \( \Delta \setminus N_{\omega} \), so are \( \tilde{\alpha}_\omega \) and \( \tilde{\beta}_\omega \). We still need to examine their behavior near \( N_{\omega} \). On a small neighborhood \( N_{\omega}(\delta) \) of \( N_{\omega} \), \( \mu_\eta' \equiv 0 \), so \( \tilde{\alpha}_\omega \equiv 0 \) and \( \tilde{\beta}_\omega \equiv 1 \). Hence \( \tilde{\alpha}_\omega \) and \( \tilde{\beta}_\omega \) are \( C^\infty \) in the neighborhood of \( \partial N_{\omega} \). Consequently they are \( C^\infty \) on all of \( \Delta \).

The functions \( \tilde{\alpha}_\omega \) and \( \tilde{\beta}_\omega \) satisfy the following relations.

\[ \mu_\eta \cdot \tilde{\alpha}_\omega = \mu_\eta \cdot \alpha_\omega, \quad \mu_\eta \cdot \tilde{\beta}_\omega = \mu_\eta \cdot \beta_\omega \] on \( \Delta \setminus N_{\omega}(\delta) \)

and \[ \tilde{\beta}_\omega > 0. \]

The previous inequality can be shown as follows.

Since \( \tilde{\beta}_\omega = 1 \) in \( N_{\omega}(\delta) \) and \( \tilde{\beta}_\omega = 0 + 1 - 0 = 1 \) in the neighborhood of \( \Delta \setminus P \), we only need to consider on the set \( P \setminus N_{\omega}(\delta) \). But in the set \( P \setminus N_{\omega}(\delta), \mu_\eta \neq 0 \).
(see the properties of $\mu_\eta$ given in the previous section), so $\mu_{\eta'} = 1$ and
\[ \tilde{\beta}_\omega = 1 - \beta_\omega + 1 = \beta_\omega > 0. \]
The last inequality follows from the hypothesis (6.8).

Let
\[ a = -\tilde{\alpha}_\omega / \tilde{\beta}_\omega, \quad b = \left( \frac{\partial \tilde{\beta}_\omega}{\partial x} - \frac{\partial \tilde{\alpha}_\omega}{\partial y} \right) \frac{1}{\tilde{\beta}_\omega} \]
and for $(x, y)$ and $(x, y_0) \in \Delta$, set
\[ y^* = y - y_0. \]

For each pair $(y_0, \omega)$, the ordinary differential equation
\[ dy^*/dx(x, y_0, \omega) = a(x, y^* + y_0, \omega), \text{ where } y^*(0, y_0, \omega) = 0 \]
has exactly one solution. Moreover $y^*(x, y_0, \omega)$ is continuous in $\Delta \times F$ and it is differentiable in $x$ and $y_0$ (see Petrowski [34, p. 70]). The same is true for the function
\[ u^*(x, y_0, \omega) = \int_0^x b(t, y_0 + y^*(t, y_0, \omega), \omega) dt. \]

The mapping $g : \Delta \times F \to \Delta \times F$, given by
\[ g(x, y_0, \omega) = (x, y_0 + y^*(x, y_0, \omega), \omega) \]
\[ = (x, y(x, y_0, \omega), \omega), \]
is continuous in $\Delta \times F$ and bijective, therefore it is a homeomorphism by the invariance of domain. Then we have

**Lemma 6.7.** The function
\[ u = u^* \circ g^{-1} : \Delta \times F \to \mathbb{R} \]
is the solution of equation
\[ \frac{\partial u}{\partial x}(x, y, \omega) + a(x, y, \omega) \frac{\partial u}{\partial y}(x, y, \omega) = b(x, y, \omega), \]
\[ u(0, y, \omega) = 0 \text{ for } (0, y, \omega) \in \Delta \times F. \]
and is $C^\infty$ in $x$ and $y$ for a fixed $\omega \in F$. 
Proof. Denote by $g^{-1} := (g_1^{-1}, g_2^{-1}, g_3^{-1})$ the inverse of $g$, where $g$ is given in equation (6.26), then we have

$$\frac{\partial u}{\partial x} = \langle \nabla u^* \circ g^{-1}, \frac{dg^{-1}}{dx} \rangle$$

$$= \langle \left( \frac{\partial u^*}{\partial g_1} \circ g^{-1}, \frac{\partial u^*}{\partial g_2} \circ g^{-1}, \frac{\partial u^*}{\partial g_3} \circ g^{-1} \right), \left( \frac{\partial g_1^{-1}}{\partial x}, \frac{\partial g_2^{-1}}{\partial x}, \frac{\partial g_3^{-1}}{\partial x} \right) \rangle$$

$$= \left( \frac{\partial u^*}{\partial x} \right) \circ g^{-1} + \left[ \left( \frac{\partial u^*}{\partial g_2} \right) \circ g^{-1} \right] \cdot \frac{\partial g_2^{-1}}{\partial x} + \left[ \left( \frac{\partial u^*}{\partial g_3} \right) \circ g^{-1} \right] \cdot \frac{\partial g_3^{-1}}{\partial x}$$

$$(6.28) = \frac{\partial u^*}{\partial x} \circ g^{-1} + \left[ \frac{\partial u^*}{\partial g_2} \circ g^{-1} \right] \cdot \frac{\partial g_2^{-1}}{\partial x} + \left[ \frac{\partial u^*}{\partial g_3} \circ g^{-1} \right] \cdot \frac{\partial g_3^{-1}}{\partial x}$$

$$= \frac{\partial}{\partial x} \int_0^x b(t, y_0 + y^*(t, y_0, \omega), \omega) dt \circ g^{-1} + \left[ \frac{\partial u^*}{\partial g_2} \circ g^{-1} \right] \cdot \frac{\partial g_2^{-1}}{\partial x}.$$

Since $g(x, y_0, \omega) = (x, y_0 + y^*(x, y_0, \omega), \omega) = (x, y, \omega)$, let the inverse of $g$ be

$$g^{-1} : (x, y, \omega) \mapsto (x, f(x, y, \omega), \omega),$$

then we obtain

$$g_2^{-1}(x, y, \omega) = f(x, y, \omega)$$

and $f$ satisfies the relation

$$f(x, y, \omega) + y^*(x, f(x, y, \omega), \omega) = y$$

since $g(g^{-1}(x, y, \omega)) = (x, y, \omega)$. Therefore, using equation (6.25), we get

$$\frac{\partial}{\partial x} \int_0^x b(t, y_0 + y^*(t, y_0, \omega), \omega) dt \circ g^{-1}(x, y, \omega)$$

$$= \frac{\partial}{\partial x} \int_0^x b(t, f(t, y, \omega) + y^*(t, f(t, y, \omega), \omega) dt$$

$$= b(x, f + y^*, \omega) = b(x, y, \omega)$$

and

$$\frac{\partial f}{\partial x} + \frac{\partial y^*}{\partial y_2} \cdot \frac{\partial f}{\partial x} + \frac{\partial y^*}{\partial x} = 0. \quad (6.32)$$

But from equation (6.24),

$$\frac{\partial y^*}{\partial x}(x, f(x, y, \omega), \omega) = a(x, f(x, y, \omega) + y^*(x, f(x, y, \omega), \omega), \omega) = a(x, y, \omega),$$
so

\begin{equation}
\frac{\partial g^{-1}}{\partial x} = \frac{\partial f}{\partial x} = \frac{(-a(x,y,\omega))}{1 + \frac{\partial y^*}{\partial g^{-1}}}.
\end{equation}

Also we obtain

\begin{equation}
\frac{\partial g^{-1}}{\partial y} = \frac{\partial f}{\partial y}.
\end{equation}

We plug (6.31) and (6.33) into (6.28) to get

\begin{equation}
\frac{\partial u}{\partial x} = b(x,y,\omega) + \frac{\partial}{\partial g^{-1}} \int_0^x b(t,f + y^*(t,f,\omega),\omega)dt \circ g^{-1} \cdot \left(\frac{(-a(x,y,\omega))}{1 + \frac{\partial y^*}{\partial g^{-1}}}\right).
\end{equation}

But on the other hand, from equations (6.34) and (6.30), we have

\begin{equation}
\frac{\partial g^{-1}}{\partial y} = \frac{\partial f}{\partial y}
\end{equation}

and

\begin{equation}
f(x,y,\omega) + y^*(x,f(x,y,\omega),\omega) = y.
\end{equation}

Differentiate the above equation to obtain

\begin{equation}
\frac{\partial f}{\partial y} + \frac{\partial y^*}{\partial g^{-1}} \cdot \frac{\partial f}{\partial y} = 1,
\end{equation}

therefore

\begin{equation}
\frac{\partial f}{\partial y} = 1 \left(1 + \frac{\partial y^*}{\partial g^{-1}}\right).
\end{equation}

Use (6.25) and the above to obtain

\begin{equation}
\frac{\partial}{\partial g^{-1}} \int_0^x b(t,f + y^*(t,f,\omega),\omega)dt \circ g^{-1} \cdot \left(1 \left(1 + \frac{\partial y^*}{\partial g^{-1}}\right)\right)
\end{equation}

\begin{equation}
= \frac{\partial (u^* \circ g^{-1})}{\partial y} \cdot \frac{\partial f}{\partial y}
\end{equation}

\begin{equation}
= \frac{\partial}{\partial y} (u^* \circ g^{-1}) = \frac{\partial u}{\partial y} (x,y,\omega)
\end{equation}

The last equality follows from our hypothesis.
Consequently, from equations (6.35) and (6.36) we get

\[ \frac{\partial u}{\partial x}(x, y, \omega) = b(x, y, \omega) - a(x, y, \omega) \frac{\partial u}{\partial y}(x, y, \omega). \]

Finally we obtain

\[ (6.37) \quad \frac{\partial u}{\partial x}(x, y, \omega) + a(x, y, \omega) \frac{\partial u}{\partial y}(x, y, \omega) = b(x, y, \omega). \]

The \( C^\infty \) part is obvious because \( u = u^* \circ g^{-1} \) is the composition of two infinitely differentiable functions. \( \square \)

From the preceding arguments \( u \) is continuous and bounded on \( \Delta \times F \), i.e., for some constant \( u_0 \),

\[ |u(x, y, \omega)| \leq u_0 \text{ for all } (x, y, \omega) \in \Delta \times F. \]

In \( \Delta \), if we let \( \varrho = e^{u_0 - u} \left( \tilde{\alpha}_\omega dx + \tilde{\beta}_\omega dy \right) \), then \( \varrho \) is closed in \( \Delta \), hence it is exact in \( \Delta \).

Therefore, there is a function \( k \) which is continuous on \( \Delta \times F \) and it is differentiable in \( x \) and \( y \) such that

\[ \varrho = dk(x, y, \omega). \]

As a final auxiliary function, recall the real-valued \( C^\infty \)-function \( \nu_\eta(x) \) for \( \eta < \frac{1}{16} \) given in Section 6.3:

1. \( |\dot{\nu}_\eta(x)| \leq 1, \)
2. \( \nu_\eta(x) = \begin{cases} x & \text{for } \eta - 1 \leq x \leq 1 - \eta \\ 2 - x & \text{for } 1 + \eta \leq x \leq 3 - \eta, \end{cases} \)
3. \( \nu_\eta(x + 4) = \nu_\eta(x). \)

Let \( N_F \) be an integer to be determined later in this section and let \( \epsilon \) be the constant given in Theorem 2.1. For

\[ N > N_F + \frac{1}{\epsilon} \cdot \max_{z \in P} |\lambda(z)|, \]

let

\[ (6.38) \quad h(x, y, \omega, N) = \frac{1}{N} \lambda(x, y) \nu_\eta(x, y, \omega) e^{u(x, y, \omega) - u_0} \cdot \nu_\eta(N \cdot k(x, y, \omega)). \]

Then \( h \) is a \( C^\infty \)-function on \( P \) and continuous in \( \omega \in F \). For \( \omega \) fixed, we obtain

\[ (6.39) \quad dh^2 = \lambda^2 \cdot \mu_\eta^2 \cdot \dot{\nu}_\eta^2 \cdot (\tilde{\alpha}_\omega dx + \tilde{\beta}_\omega dy)^2 + o\left( \frac{1}{N} \right) |dz|^2. \]
In the Section 6.5, we will check the conditions of the Garsia’s Continuity Lemma 4.1. So we still want to show the area of the exceptional set

\[(6.40) \quad A = \{ (x, y) \in P \mid \mu_\eta(x, y, \omega) \cdot \nu_\eta^2 (N \cdot k(x, y, \omega)) \neq 1 \}\]

can be made arbitrarily small.

Since

\[ \text{area } \{ (x, y) \in P \mid \mu_\eta(x, y, \omega) \neq 1 \} < \eta/2, \]

we only need to determine the area of the set

\[(6.41) \quad A_1 := \{ (x, y) \in P \mid |k(x, y, \omega) - \frac{1}{N}| \leq \frac{\eta}{N} \mod \frac{2}{N} \} \]

To compute the area of \( A_1 \), first of all, we see the mapping

\[(6.42) \quad \Phi_\omega(x, y) = (x, k(x, y, \omega)) : P \to \mathbb{C} \]

is one-to-one because

\[ \det(D\Phi_\omega) = \frac{\partial k}{\partial y} \]

and

\[(6.43) \quad \frac{\partial k}{\partial y}(x, y, \omega) = e^{u_0 - u} \cdot \tilde{\beta}_\omega > 0. \]

Since \( \det(D\Phi_\omega) = e^{u_0 - u} \cdot \tilde{\beta}_\omega > 0 \) is continuous on \( P \times F \), there exists a \( \sigma > 0 \) such that

\[(6.44) \quad e^{u_0 - u} \cdot \tilde{\beta}_\omega \geq \sigma, \text{ for all } (x, y, \omega) \in P \times F. \]

Suppose \( |k(x, y, \omega)| \leq k_0 \), then \( \Phi_\omega(A_1) \) lies in a collection of horizontal strips in the rectangle \([-1, 1] \times [-k_0, k_0]\).

From the definition (6.41) of the set \( A_1 \), we know that

\[ \left( \frac{1}{N} + \frac{2m}{N} \right) - \frac{\eta}{N} \leq k \leq \left( \frac{1}{N} + \frac{2m}{N} \right) + \frac{\eta}{N} \]

and we only need to consider those values of \( m \) for which

\[ \frac{1}{N} + \frac{2m}{N} - \frac{\eta}{N} \leq k_0. \]

For sufficiently large \( N \),

\[ \frac{1}{N} + \frac{\eta}{N} \leq k_0 \]
so that
\[ \frac{2m}{N} \leq k_0 + \frac{\eta}{N} - \frac{1}{N} \]
\[ \leq k_0 + k_0, \]
therefore
\[ m \leq Nk_0. \]

Finally we obtain
\[ \text{area } (\Phi_\omega(A_1)) \leq (\text{width of a strip}) \cdot (\text{possible values of } m) \]
\[ = \frac{2\eta}{N} \cdot 2Nk_0 \]
\[ = \eta \cdot 4k_0. \]

Therefore
\[ \text{(6.45)} \]
\[ \text{area } (A_1) = \int_{\Phi_\omega(A_1)} |\det(D\Phi^{-1}_\omega)| \, dx dy \leq \frac{1}{\sigma} \text{area}(\Phi_\omega(A_1)) < k_F \cdot \eta, \]
if \( k_F > 4k_0/\sigma \).

We obtain
\[ \text{(6.46)} \]
\[ \text{area } A < \frac{\eta}{2} + k_F \cdot \eta. \]

Here we take \( N_F > 4(k_F \sigma - 4k_0) \) so that the inequality (6.47) is valid for
\[ N > N_F + (1/\epsilon) \cdot \max_{z \in P} |\lambda(z)|. \]

6.5. Comparison of the Metrics \((d\mathcal{S}^\omega)^2\) and \(d\tau^2_\omega\). We will need the following two lemmas. Together with equation (5.2), they are used to compute the dilatation \( K_\chi \) of the mapping \( \chi = \mathcal{S}^\omega \circ (id_\omega)^{-1} \) and to prove the continuity of \((d\mathcal{S}^\omega)^2\) in \((z, \omega) \in \tilde{S} \times \mathcal{T}^c(\omega_0)\).

**Lemma 6.8.** Given \( h = h(x, y, \omega, N) \) as in equations (6.10) and (6.38), and \( \gamma_\omega ds^2_\omega \) as in (6.9),
\[ |dh^2 + \lambda^2 |dz|^2 - \gamma_\omega ds^2_\omega| \leq \begin{cases} R(\eta; N)ds^2_\omega & \text{on } P - A, \, \omega \in F \\
R(\eta; N)ds^2_\omega & \text{on } A, \, \omega \in F, \end{cases} \]
where the area of \( A \) is given by inequality (6.20) (if \( g = 1 \)) or (6.47) (if \( g > 1 \)). The inequalities are valid for \( N > N_F + (1/\epsilon) \cdot \max_{z \in P} |\lambda(z)| \). \( N_F \) is a constant depending on the compact set \( F \). For each fixed \( \eta, R(\eta; N) \) can be made small as \( N \to \infty \) and \( R(\eta; N) \) is some constant which is bounded as a function of \( N \).
Proof. (1) In genus $g = 1$, use equations (6.11) and (6.9) to get
\[
|dh^2 + \lambda^2|dz|^2 - \gamma_\omega ds^2_\omega|
\]
(6.48) \[= \left| \lambda^2 \cdot (\alpha_\omega dx + \beta_\omega dy)^2 (\mu_\eta^2(x) \cdot \mu_\eta^2(y) \cdot \dot{\nu}_\eta^2 - 1) + o\left(\frac{1}{N}\right)|dz|^2 \right| \]

On $P - A$, we have $\mu_\eta(x) = \mu_\eta(y) = \dot{\nu}_\eta(x) = 1$ (see equation (6.13) and Lemma 6.2), so the right hand side of equation (6.48) becomes
\[
\left| o\left(\frac{1}{N}\right)|dz|^2 \right| \leq R(\eta; N) ds^2_\omega
\]
for some small constant $R(\eta; N)$.

On $A$, we have $\mu_\eta^2(x) \cdot \mu_\eta^2(y) \neq 1$ and $\dot{\nu}_\eta^2 \neq 1$ (see equation (6.13) and Lemma 6.2), so the right hand side (RHS) of equation (6.48) becomes
\[
\text{RHS} \leq \gamma_\omega \left[ \frac{\mu_\eta^2(x) \mu_\eta^2(y) \dot{\nu}_\eta^{2} - 1 + o\left(\frac{1}{N}\right)}{ds^2_\omega} \right]
\]
(6.49) \[\leq \tilde{R}(\eta; N) ds^2_\omega \]
for some constant $\tilde{R}(\eta; N)$ which is not necessarily very small.

(2) In genus $g > 1$, the computation is exactly the same, however the equations that justify it are equations (6.39) and (6.9). Use them to obtain
\[
|dh^2 + \lambda^2|dz|^2 - \gamma_\omega ds^2_\omega|
\]
(6.50) \[= \left| \lambda^2 \cdot (\alpha_\omega dx + \beta_\omega dy)^2 (\mu_\eta^2(x) \cdot \dot{\nu}_\eta^2 - 1) + o\left(\frac{1}{N}\right)|dz|^2 \right| .
\]

On $P - A$, we have $\mu_\eta^2 = \dot{\nu}_\eta^2 = 1$ (see equation (6.40) and property (2) of the function $\mu_\eta$), so the right hand side of equation (6.50) becomes
\[
\left| o\left(\frac{1}{N}\right)|dz|^2 \right| \leq R(\eta; N) ds^2_\omega
\]
for some small constant $R(\eta; N)$.

On $A$, since $\mu_\eta^2 \cdot \dot{\nu}_\eta^2 \neq 1$, the right hand side (RHS') of equation (6.50) becomes
\[
\text{RHS'} \leq \gamma_\omega \left[ (\mu_\eta^2 \cdot \dot{\nu}_\eta^2 - 1) + o\left(\frac{1}{N}\right) \right] ds^2_\omega \leq \tilde{R}(\eta; N) ds^2_\omega
\]
for some constant $\tilde{R}(\eta; N)$ which is not necessarily very small. \[\square\]

**Lemma 6.9.** Given $h(x, y, \omega, N)$ as in equations (6.10) and (6.38), the metric of the deformed surface $S^\omega := S_{h(x, \omega)}$, defined by
\[
\mathcal{S}^\omega(x, y) = X(x, y) + h(x, y; \omega, N)\tilde{\Gamma}(X(x, y)) + r(h(x, y)^2),
\]
satisfies the inequality
\[ |(d\mathcal{S}^\omega)^2 - dh^2 - dX^2| \leq c(\eta; N)ds_\omega^2 \]
for each fixed \( \eta \) and \( \omega \in F \), where \( c(\eta; N) \to 0 \) as \( N \to \infty \).

**Proof.** From equation (6.4), we obtain
\[ (d\mathcal{S}^\omega)^2 = dX^2 + dh^2 + o(h)|dz|^2 \]
for the metric \((d\mathcal{S}^\omega)^2\).

Since \( h \) can be made arbitrarily small, as \( N \to \infty \), the remainder term \( o(h) \to 0 \). Therefore, we get
\[ |(d\mathcal{S}^\omega)^2 - dh^2 - dX^2| = |o(h)||dz|^2 \leq c(\eta; N)ds_\omega^2, \]
where the constant \( c(\eta; N) \) can be made arbitrarily small as \( N \to \infty \). \( \square \)

Application of Lemma 6.8 and Lemma 6.9 gives the following estimates for the \( \epsilon \)-normal deformation \( S^\omega \) of \( S \).

**Lemma 6.10.** Assume that \( h(x, y, \omega, N) \) is given by either equation (6.10) or (6.38) and that supremum and infimum are taken over all directions at a point \( z \). Then the metric of the deformed surface \( S^\omega := S_{h(\cdot, \omega)} \), defined by the map \( S^\omega(x, y) \) as given in Lemma 6.9, satisfies the relations:

1. \( (\sup ((d\mathcal{S}^{\omega_m})^2/(d\mathcal{S}^\omega)^2)) / (\inf ((d\mathcal{S}^{\omega_m})^2/(d\mathcal{S}^\omega)^2)) \to 1 \) as \( \omega_m \to \omega \).
2. \( K^2 \leq \begin{cases} 1 + c_1(\eta; N) & \text{on } P - A \\ 4c_2(\eta; N) & \text{on } A \end{cases} \)
   if \( \omega \in F \), where the constant \( c_1 \) can be made arbitrarily small for each fixed \( \eta \) and for sufficiently large \( N \). \( c_2 \) is some constant which is not necessarily small. The area of \( A \) is given by the inequality (6.20) (if \( g = 1 \)) or (6.47) (if \( g > 1 \)).

**Proof.** Since \( h(x, y, \omega, N) \) is \( C^\infty \) on \( P \) and continuous in \( \omega \in F \),
\[ h(x, y, \omega_m, N) \to h(x, y, \omega, N) \text{ as } \omega_m \to \omega. \]

By equations (6.11) and (6.39), we see that \( dh \) is continuous on \( P \times F \) because every function used in \( dh \) is continuous on \( P \times F \), therefore we get
\[ dh(x, y, \omega_m, N) \to dh(x, y, \omega, N) \text{ as } \omega_m \to \omega. \]
Therefore the metric \((d\mathcal{S}^\omega)^2\), given in equation (6.52), is continuous on the compact set \(P \times F\), that is,

\[
\left( d\mathcal{S}^{\omega_m} \right)^2 \rightarrow \left( d\mathcal{S}^\omega \right)^2 \quad \text{as} \quad \omega_m \rightarrow \omega.
\]

From the above observation, property (1) follows immediately.

Using the inequalities in Lemma 6.8 and Lemma 6.9, we derive the following estimates.

On \(A\),

\[
\begin{align*}
\sup \left( \frac{d\mathcal{S}^\omega}{ds_z^2} \right)^2 & = \sup \left( \frac{d\mathcal{S}^\omega}{ds_z^2} - dh^2 - dX^2 + dh^2 + dX^2 \right) \\
& \leq \sup \frac{dX^2 + dh^2}{ds_z^2} + \sup \frac{(d\mathcal{S}^\omega)^2 - dh^2 - dX^2}{ds_z^2} \\
& \leq \sup \frac{dX^2 + dh^2}{ds_z^2} + c(\eta; N) \\
& \leq \gamma_\omega + \hat{R}(\eta; N) + c(\eta; N)
\end{align*}
\]

which is uniformly bounded.

On \(P - A\),

\[
\begin{align*}
\sup \left( \frac{d\mathcal{S}^\omega}{ds_z^2} \right)^2 & = \sup \left( \frac{dX^2 + dh^2 + (d\mathcal{S}^\omega)^2 - dh^2 - dX^2}{ds_z^2} \right) \\
& \leq \sup \frac{dX^2 + dh^2}{ds_z^2} + \sup \frac{(d\mathcal{S}^\omega)^2 - dh^2 - dX^2}{ds_z^2} \\
& \leq \sup \frac{dX^2 + dh^2}{ds_z^2} + c(\eta; N) \\
& \leq \gamma_\omega + R(\eta; N) + c(\eta; N).
\end{align*}
\]

Recall that \(dX^2 = \lambda^2(z)|dz|^2\) from the conformality of the parameter \(X\).
On $A$,
\[
\inf \frac{(d\mathcal{S}_\omega)^2}{ds_\omega^2} \geq \inf \frac{dX^2 + (d\mathcal{S}_\omega)^2 - dh^2 - dX^2}{ds_\omega^2} \\
\geq \inf \frac{dX^2}{ds_\omega^2} - c(\eta; N)
\]
\[
= \inf \frac{\lambda^2|dz|^2}{\lambda^2|dz| + \Psi_\omega d\bar{z}|^2} - c(\eta; N)
\]
\[
\geq \inf_{\omega \in M} \frac{1}{(1 + \|\Psi_\omega\|_\infty)^2} - c(\eta; N)
\]
\[
> \frac{1}{4} - c(\eta; N)
\]
(6.55)

since $\lambda^2(z)|dz + \Psi_\omega(z)d\bar{z}|^2 \leq \lambda^2(z)(1 + \|\Psi_\omega\|_\infty^2)|dz|^2$ by inequality (6.3).

On $P - A$,
\[
\inf \frac{(d\mathcal{S}_\omega)^2}{ds_\omega^2} = \inf \left[ \frac{\gamma_\omega ds_\omega^2 + dh^2 + \lambda^2|dz|^2 - \gamma_\omega ds_\omega^2}{ds_\omega^2} \\
+ \frac{(d\mathcal{S}_\omega)^2 - dh^2 - dX^2}{ds_\omega^2} \right]
\]
\[
\geq \inf \frac{\gamma_\omega ds_\omega^2}{ds_\omega^2} - \sup \frac{dh^2 + \lambda^2|dz|^2 - \gamma_\omega ds_\omega^2}{ds_\omega^2} \\
- \frac{(d\mathcal{S}_\omega)^2 - dh^2 - dX^2}{ds_\omega^2} \\
\geq \gamma_\omega - R(\eta; N) - c(\eta; N).
\]

Inequality (2) then follows directly.

6.6. **Proof of the Theorem 2.1.** So far we have checked every condition we need in the hypotheses of Garsia’s Continuity Lemma 4.1 for some compact set $F$ in $Q_1(S)$. Therefore if we take $\epsilon = \frac{1}{2} \min \{1 - \|\omega_0\|, \|\omega_0\|\}$ and $F = \overline{B}_\epsilon(\omega_0) \subset Q_1(S) \setminus \{0\}$, then we may now complete the proof of Theorem 2.1 by the arguments given in Section 5.2 together with Lemma 6.1.

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(Seok-Ku Ko) DEPARTMENT OF APPLIED MATHEMATICS, KONKUK UNIVERSITY, CHUNGJU, CHUNGKUK 380-701, KOREA
E-mail address: seokko@apmath.kku.ac.kr, seokko@kku.ac.kr