TEICHMÜLLER THEORY AND ITS APPLICATIONS

SEOKKU KO

1. Introduction

The theory of Teichmüller spaces studies the different conformal structures on a Riemann surface. After the introduction of quasiconformal mappings into the subject, the theory can be said to deal with classes consisting of quasiconformal mappings of a Riemann surface which are homotopic modulo conformal mappings.

It was Teichmüller who noticed the deep connection between quasiconformal mappings and function theory. He also discovered that the theory of Teichmüller spaces is intimately connected with quadratic differentials. Teichmüller ([36], [37]) proved that on a compact Riemann surface of genus greater than one, every holomorphic quadratic differential determines a quasiconformal mappings which is a unique extremal in its homotopy class in the sense that it has the smallest deviation from conformal mappings. He also showed that all extremals are obtained in this manner. It follows that the Teichmüller space of compact Riemann surface of genus \( g > 1 \) is homeomorphic to the euclidean space \( \mathbb{R}^{6g-6} \).

Teichmüller’s proofs, often sketchy and intermingled with conjectures, were put on a firm basis by Ahlfors [5], who also introduced a more flexible definition for quasiconformal mappings.

Another approach to the Teichmüller theory, initiated by Bers in the early sixties, leads to quadratic differentials in an entirely different manner. This method is more general, in that it can also be applied to non-compact Riemann surfaces. The quadratic differentials are now Schwarzian derivatives of conformal extensions of quasiconformal mappings considered on the universal covering surface, the extensions being obtained by use of the Beltrami equation.

In 1882, Gauss showed that if you have any orientable surface \( S \), then for \( p \in S \), and for small neighborhood \( N \) of \( p \), there exists a conformal map on \( N \) to another surface.

2. Conformal structures in the plane

Since Teichmüller theory has to do with putting different conformal structures on the same surface, we want to begin by describing these structures as conveniently as we can. Any Riemannian metric on a plane region \( D \) can be written in the form

\[
ds^2 = E \, dx^2 + 2F \, dx \, dy + G \, dy^2,
\]
where $E, F$ and $G$ are real-valued functions in $D$ such that $E$ and $EG - F^2$ are positive everywhere. However it is better for our purposes to use complex notation and write

$$ds = \lambda(z)|dz + \mu(z)d\bar{z}|,$$

where $\lambda(z)$ is a positive function in $D$, and $\mu(z)$ is a (Borel) measurable complex-valued function with $|\mu(z)| < 1$. It is well known and easy to verify that every metric can be written uniquely in the form (1).

Now suppose that $w : D \to D^*$ is a sense preserving diffeomorphism of $D$ onto $D^*$. Then $w$ is a conformal map (with respect to the given metrics) if and only if $|dz + \mu(z)d\bar{z}|$ is proportional to

$$|d\zeta + \mu^*(\zeta)d\bar{\zeta}| = |dw(z) + \mu^*(w(z))d\bar{w}(z)|$$

at all $z$ in $D$. For example, if $ds^* = |d\zeta|$ is the Euclidean metric in $D^*$, then $w$ is conformal if and only if it is a solution of the equation

$$w_z = \mu(z)w_z$$

in $D$, where

$$w_z = \frac{1}{2} \left( \frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right),$$

$$w_{\bar{z}} = \frac{1}{2} \left( \frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right).$$

**Definition 2.1.** (2) is the classical Beltrami equation, which goes over into the Cauchy-Riemann equation for $u$ and $v$ ($\frac{\partial w}{\partial \bar{z}} = 0$ with $w = u + iv$) if $\mu(z) = 0$. Note that $\lambda$ does not enter into (2), as was expected.

We call the metrics $ds$ and $ds^*$ on $D$ **conformally equivalent** if and only if $ds$ and $ds^*$ are proportional at every point of $D$, or in other words the identity map of $D$ onto itself is conformal (with respect to the given metrics $ds$ and $ds^*$). A conformal equivalence class of metrics is called a **conformal structure**. It is clear from the representation (1) that the conformal structures on $D$ are in one-to-one correspondence with the complex valued functions $\mu(z)$ in $D$ with $|\mu(z)| < 1$. In fact each such function corresponds to the conformal structure determined by the metric (1).

3. Quasiconformal mappings

We call the solution of (2) $\mu$-**conformal** functions which are assumed to be continuous and to have distributional derivatives which are measurable and locally square integrable functions. The function $\mu$ is called the **Beltrami coefficient** of the mapping $w$.

**Definition 3.1.** $\mu(z) = \frac{w_{\bar{z}}}{w_z}$ is called the complex dilatation of $w$, where $\|\mu\|_\infty < 1$.

Gauss solved that there exists a local homeomorphic solution for the Beltrami equation (2) and later he solved the Beltrami equation for $\mu$ being real analytic (which means that $\mu$ can be expressed as a power series). In 1938, Morrey showed that if $\mu$ is a measurable function which
permits discontinuity at some points, then there exists a solution whose partial derivatives are not necessarily continuous to the Beltrami equation (2). This extends the local results of Gauss to the global results. We here note that the any solution of the Beltrami equation satisfies the Hölder condition.

**Definition 3.2.** A sense-preserving homeomorphism $w : D \to D'$ is called quasiconformal if it is $\mu$-conformal for some $\mu$ with $\|\mu\|_\infty = \text{ess sup} |\mu(z)| < 1$. If so, the number $K(w) = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)$ is called the dilatation of $w$. In this case we say that $w$ is $K$-quasiconformal ($K$-qc).

If $w$ is a diffeomorphism, $K(w)$ is the supremum of the ratios of the major to the minor axes of infinitesimal ellipses into which $w$ takes infinitesimal circles in the domain considered.

We list some useful properties of quasiconformal mappings.

**Proposition 3.1.** Let $f : D \to D'$ be $K$-qc, then

(a) $f$ is differentiable a.e.
(b) $|f_z| > 0$ a.e.
(c) $\text{mes}(f(E)) = \int \int_E (|f_1|^2 - |f_2|^2) dx dy$ for all measurable sets $E \subset D$.
(d) $f^{-1} : D' \to D$ is $K$-qc. If $g : D' \to D''$ is $K''$-qc, then $g \circ f$ is $KK''$-qc in $D$.
(e) If $K = 1$, then it is conformal (Weyl’s lemma).

The Beltrami equation (2) has the quasiconformal solution by the following existence theorem which is due to Ahlfors-Bers.

**Theorem 3.1** (Ahlfors-Bers). Let $S$ be a Riemann sphere $\hat{\mathbb{C}}$ and $\mu$ be a measurable function defined for all $z \in \mathbb{C}$ and $\|\mu\|_\infty < 1$, then there exists a quasiconformal mapping $w^\mu$ of $\mathbb{C}$ onto itself fixes zero and one with $\mu(w^\mu(z)) = \mu(z)$ and satisfying the Beltrami equation (2). The mapping is unique modulo left composition with a M"obius transformation. Further, if $\mu$ depends holomorphically, real analytically, or $C^\infty$ on some parameters, then so does $w^\mu(z)$. Explicitly we have

$$w^\mu(z) = z + P\mu(z) + o(\|\mu\|),$$

where $o(\|\mu\|) \to 0$ is uniform as $\mu \to 0$ on compact subsets of $\mathbb{C}$ and $P\mu$ is given by

$$P\mu(z) = -\frac{z(z - 1)}{\pi} \int \int \frac{\mu(\zeta)}{\zeta(\zeta - 1)(\zeta - z)} d\xi d\eta,$$

and $\zeta = \xi + i\eta$.

**Corollary 3.1.** There exists a unique quasiconformal mapping $w_\mu$ of $H$ onto itself for fixed $\mu$ which is continuous on $\hat{\mathbb{R}}$ and keeps 0, 1, $\infty$.

**Proof** Let $w_\mu = w^\nu|H$, where $\nu(\bar{z}) = \overline{\nu(z)}$, $\nu|H = \mu$. The mapping $w_\mu$ has properties similar to those of $w^\mu$, except that the dependence on $\mu$ is real analytic rather than holomorphic.

Q.E.D.
Above map sends real axis to real axis if and only if it satisfies the boundary condition,
\[
\frac{1}{M} < \frac{|w_\mu(x + h) - w_\mu(x)|}{|w_\mu(x) - w_\mu(x - h)|} < M
\]
for some positive constant \( M \).

4. Conformal structures on a surface and the uniformization theorem

Let \( S \) be any smooth (class \( C^\infty \)) surface. Once again we call two smooth Riemannian metrics on \( S \) conformally equivalent if they are proportional at every point, and we call an equivalent class of metrics a \textit{conformal structure} on \( S \).

An oriented surface \( S \) with a given conformal structure is called a \textit{Riemann surface}. A Riemann surface is often defined to be a connected one-dimensional complex manifold, but the above definition is readily seen to be equivalent. In fact if \( S \) is oriented and has a given conformal structure, then the sense-preserving conformal maps from open sets of \( S \) into the complex plane \( \mathbb{C} \) (with its natural Euclidean metric) form a complex analytic atlas on \( S \). Conversely, every connected one-dimensional complex manifold \( S \) has a natural orientation, and it is a classical fact, following easily from the uniformization theorem (which will be introduced in the next), that \( S \) has a Riemannian metric such that the complex coordinate functions on \( S \) are sense preserving conformal maps into \( \mathbb{C} \).

A conformal map between Riemann surfaces is a sense-preserving diffeomorphism which is conformal with respect to the given conformal structures. The Riemann surfaces \( S \) and \( S' \) are called \textit{equivalent} if there is a conformal map of \( S \) onto \( S' \). Notice that this is a much weaker equivalence relation than conformal equivalence of metrics. Two Riemannian metrics on \( S \) are \textit{conformally equivalent} if the identity map on \( S \) is conformal (with respect to the given metrics). They define equivalent Riemann surfaces if there is some sense preserving diffeomorphism \( f : S \rightarrow S \) which is conformal.

By the works of Rüedy [32],[33],[34] and Garsia [18], we may always think of a Riemann surface as an oriented, sufficiently smooth, surface embedded into Euclidean space, provided we agree that two surfaces are identical qua Riemann surfaces if there is a conformal bijection between them. Recently author showed that we may think of a Riemann surface as an oriented, sufficiently smooth, surface embedded into orientable Riemannian manifold of dimension more than 3 (later in this talk we will see this) [22].

Let
\[
\mathcal{M} \text{ö}b_\mathbb{C} = \left\{ \gamma(z) : \gamma(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \quad a, b, c, d \in \mathbb{C} \right\}.
\]

\( G < \mathcal{M} \text{ö}b_\mathbb{C} \) acts \textit{properly discontinuously} (respectively \textit{discontinuously}) if the point \( x \in \hat{\mathbb{C}} \) has a neighborhood \( N \) such that \( g(N) \cap N \neq \emptyset \) implies \( g = id \) (respectively there are finitely many such elements). Let \( \Omega = \Omega(G) \) = the region of discontinuity, if \( \Omega(G) \neq \emptyset \), then we call \( G \) a \textit{Kleinian group}.

Let \( \Lambda = \Lambda(G) = \) the set of limit points. \( x \) is a \textit{limit point} for the Kleinian group \( G \) if there is a point \( z \in \Omega \) and there is a sequence \( \{ g_n \} \) of distinct elements of \( G \) with \( g_n(z) \to x \).
limit set contains either at most two points or else uncountably many points. The limit set of a Kleinian group is nowhere dense in the plane.

A Kleinian group $G$ is called Fuchsian if all its loxodromic elements are hyperbolic and $G$ leaves a disc or a half plane invariant; one can achieve by conjugation, that this be $H$, so that all $g \in G$ have real coefficients $a, b, c, d$. The limit set of a Fuchsian group acting on a disc $D$ is either the whole boundary $\partial D$ or a nowhere dense subset of $\partial D$. If the $\Lambda$ of the Fuchsian group $G$ is all of $\partial D$, then $G$ is said to be of the first kind. Otherwise $G$ is of the second kind.

A fundamental region for a Kleinian group $G$ is an open set $D$ in $\Omega$ such that its boundary (in $\Omega$) has measure zero, no two interior points of $D$ are $G$--equivalent, and every point of $\Omega$ is $G$--equivalent to a point of the closure of $D$.

The quotient $\Omega(G)/G$ has a canonical complex structure determined by the condition; the projection $\pi: \Omega \to \Omega / G$ is holomorphic.

Let $S$ be a Riemann surface of type $(g, n)$, that is, one obtained from a compact surface of genus $g$ by removing $n$ distinct points. Then we get the following celebrated uniformization theorem conjectured by Klein and by Poincaré in 1882 and proved by Poincaré and by Koebe in 1907.

**Theorem 4.1.** If $S$ is a Riemann surface of type $(g, n)$ with $2g - 2 + n > 0$, then $S$ admits an (essentially unique) representation of the form

$$S = H / G,$$

where $H$ is the upper half plane and $G$ is a torsion-free Fuchsian group, that is, a discrete group of real Möbius transformations (conformal self-mappings of $H$).

The Ahlfors-Bers existence theorem yields a short proof of the uniformization theorem.

There is a simple geometric method, going back to Poincaré, to construct a Fuchsian group $G$ such that $H / G$ is some Riemann surface of the desired type $(g, n), H / G = S$.

Let $G_0$ be a torsion-free discrete subgroup of $\text{M}_\mathbb{C}$ so that $\pi_0: H \to S_0$ be a holomorphic universal covering map with the covering group $G_0$ and therefore

$$H / G_0 = S_0.$$ 

If there is another surface $S$ of the same type as $S_0$, then there is a holomorphic universal covering map $\pi: H \to S$ with some covering group which we want to find.

Now, there is a quasiconformal homeomorphism (or diffeomorphism) $f: S_0 \to S$, and if $ds^2$ is some Riemannian metric on $S$ which respects the conformal structure of $S$, we can pull it back to $S_0$ by $f$, and obtained on $S_0$ a smooth Riemannian metric $ds_0^2$ which will, in general, not respect the conformal structure of $S_0$. But by the construction of $\pi_0$ above, we can, using this map, pull back $ds_0^2$ to a Riemannian metric on $H$ which we write in the form $\lambda^2 |dz + \mu(z)d\bar{z}|^2$. Thus we obtain in $S$ a function $\mu(z)$, which is invariant on $G_0$ with $\|\mu(z)\|_\infty < 1$. Next we let $w_\mu$ be the normalized $\mu$--conformal self-mapping of $H$ which makes the diagram commutes.
and form the group
\[ G_\mu = w_\mu G_0 w_\mu^{-1}. \]

A direct calculation shows that this group is a discrete group of Möbius transformations, mapping \( \mathbb{H} \) onto itself, thus a torsion-free Fuchsian group, and gives an isomorphism between \( G_0 \) and \( G_\mu \). Furthermore, \( G_\mu \) satisfies the condition of the covering group for \( \pi \) and it also gives a whole covering group since the diagram commutes. So, finally we have a holomorphic (and conformal) universal covering \( \pi \) with covering group \( G_\mu \). Hence \( \mathbb{H}/G_\mu = S \), as required. Here we note that
\[ \mu(z) = \mu_{S_0}(\pi_0(z)) \frac{\pi_0'(z)}{\pi_0(z)}, \quad z \in \mathbb{H}. \]

Suppose we have Fuchsian group \( G \) on \( \mathbb{H} \), and let \( \mathbb{H}/G = S \), which is a Riemann surface. Then we get another Riemann surface \( S \), which is a mirror image of \( S \), defined by \( L/G \), where \( L \) is a lower half plane. In this case local parameters are conjugate and the topologies are the same.

**Theorem 4.2.** Given 2 homeomorphic Riemann surfaces \( S_0 \) and \( S \), we can represent them by
\[ S = \mathbb{H}/G, \quad S_0 = L/G, \]
where \( G \) is a group of conformal self maps of whole plane.

5. **The Teichmüller space \( T(S) \)**

Assume that \( S \) is a Riemann surface of quasiconformal type \( (g, n) \), \( 2g - 2 + n > 0 \) unless otherwise stated. Choose a fixed Riemann surface \( S \) of type \( (g, n) \) and form a pair \( (S_1, f_1) \) consisting of a topological oriented surface \( S_1 \) of type \( (g, n) \) and an orientation-preserving quasiconformal mapping \( f_1 : S \to S_1 \). The pair \( (S_1, f_1) \) is called a marked Riemann surface.

Two marked Riemann surfaces \( (S_1, f_1) \) and \( (S_2, f_2) \) are said to be topologically equivalent if \( S_1 \) is homeomorphic to \( S_2 \) and \( f_1^{-1} \circ f_2 \) is a homeomorphism of \( S \) homotopic to the identity. They are conformally equivalent if \( f_2 \circ f_1^{-1} \) is homotopic to a conformal mapping of \( S_1 = f_1(S) \) onto \( S_2 = f_2(S) \). A homeomorphism \( f : S_1 \to S_2 \) is called a mapping of \( (S_1, f_1) \) onto \( (S_2, f_2) \) and is denoted \( f : (S_1, f_1) \to (S_2, f_2) \) if \( f \) is homotopic to \( f_2 \circ f_1^{-1} \).

\[ \begin{array}{c}
S_1 \\
\downarrow f_1 \\
S \\
\downarrow f_2 \\
S_2
\end{array} \]

\[ f_2 \circ f_1^{-1} \cong \text{conformal} \]
Definition 5.1. The Teichmüller space $T(S)$ is the set of all conformal equivalence classes of marked Riemann surfaces $(S_1, f_1)$, where $f_1 : S \to S_1$. We will also denote the equivalence class $[(S_1, f_1)]$ as $[f_1]$.

We define the modular group $\text{mod} (S)$ as the factor group of all quasiconformal selfmappings $g$ of $S$ over the normal subgroup of those homotopic to the identity modulo $\partial S$. The element of $\text{mod} (S)$ defined by a selfmapping $g$ will be denoted by $[g]$. This group acts on $T(S)$ as follows: $[g] \in \text{mod} (S)$ induces the selfmapping $[f] \mapsto [f \circ g^{-1}]$ of $T(S)$. The action is not necessarily effective; it may happen that $\text{id} \neq [g] \in \text{mod} (S)$ but $[f \circ g^{-1}] = [f]$ for all $[f] \in T(S)$. For instance, if $S = \mathbb{C} - \{0, 1\}$, then $T(S)$ is a point but $\text{mod} (S)$ has order $6$.

Given any Riemann surface $S$ of type $(g, n)$, $2g - 2 + n \geq 0$ which includes a compact torus, we define the Teichmüller distance between $[f_1], [f_2] \in T(S)$ by

$$d([f_1], [f_2]) = \inf_{h} \left\{ \frac{1}{2} \log(\sup_{z} K_h(z)) \mid h \simeq f_2 \circ f_1^{-1} \right\},$$

where $K_h(z)$, the dilatation of $h$ at $z$, is defined by

$$K_h(z) = \frac{|h_z(z)| + |h_{\bar{z}}(z)|}{|h_z(z)| - |h_{\bar{z}}(z)|},$$

and $\simeq$ denotes free homotopy.

Since the dilatation of a $K$–quasiconformal mapping is invariant under conformal transformations, this distance is well defined.

Corollary 5.1. Teichmüller space $T(S)$ is a complete metric space with the Teichmüller metric.

Proof. See Abikoff [1], Bers [8], Gardiner [16] or Nag [29]. Q.E.D.

Theorem 5.1. The Teichmüller spaces of two quasiconformally equivalent Riemann surfaces are isometrically bijective.

Proof. See Lehto [27] Q.E.D.

In 1969, Royden showed that Teichmüller distance is the same as Kobayashi distance, that is, the Teichmüller distance is independent of quasiconformal map. He proved this fact by constructing a hyperbolic metric and using generalized Schwarz lemma.

If we repeat the preceding definitions, omitting all references to ideal boundary curves, we obtain, instead of the Teichmüller space $T(S)$, the so-called reduced Teichmüller space $T^\#(S)$, and instead if the modular group $\text{mod} (S)$ the reduced modular group $\text{mod}^\# (S)$. There is an obvious canonical surjection $T(S) \to T^\#(S)$ which defines a Teichmüller metric on $T^\#(S)$ and induces an epimorphism $\text{mod} (S) \to \text{mod}^\# (S)$.

One verifies that two elements of $T(S)$, $[f_1]$ and $[f_2]$, are equivalent under the group $\text{mod} (S)$ if and only if the Riemann surfaces $f_1(S)$ and $f_2(S)$ are conformally equivalent. Hence

$$X(S) = T(S)/\text{mod} (S),$$

is the space of conformal equivalence classes (moduli) of Riemann surfaces quasiconformally equivalent to $S$. One can also show that $X(S) = T^\#(S)/\text{mod}^\#(S)$.

6. Beltrami differentials and Quasi-Fuchsian group

If $G$ is a Kleinian group, a Beltrami coefficient $\mu$ on $\mathbb{C}$ is called Beltrami coefficient on $G$ provided that

$$\mu(g(z))\overline{g'(z)} = \mu(z), \quad g \in G, \quad \text{and} \quad \mu|\Lambda = 0.$$  

Assume that this is so. Then for every $g \in G$, the function $w^\mu(g(z))$ is again a $\mu$-conformal automorphism of $\hat{\mathbb{C}}$, as is verified easily. Hence there is a Möbius transformation $g_1$ with $w^\mu \circ g = g_1 \circ w^\mu$. We conclude that

$$G^\mu = w^\mu G (w^\mu)^{-1}$$

is again a Kleinian group. The elements of $G^\mu$ depend holomorphically on the parameter $\mu$. The groups $G^\mu$ are called quasiconformal deformations of $G$.

The mapping $G \to G^\mu$ given by $g \mapsto w^\mu \circ g \circ (w^\mu)^{-1}$ is called a quasiconformal isomorphism defined by $\mu$, or a $\mu$-conformal deformation. Observe that

$$\Omega(G^\mu) = w^\mu(\Omega(G))$$

and the mapping $w^\mu : \Omega \to \Omega(G^\mu)$ induces quasiconformal mapping of the components of $\Omega(G)/G$ into the corresponding components of $\Omega(G^\mu)/G^\mu$.

If $G$ is a Fuchsian group and $\mu$ is a Beltrami coefficient for $G$, then $G^\mu$ is a quasi-Fuchsian group with fixed curve $w^\mu(\hat{\mathbb{R}})$. If $\mu$ also satisfies the symmetry condition $\mu(z) = \mu(\bar{z})$, then $G^\mu$ is again a Fuchsian group.

We define $M(G)$, the space of Beltrami differentials, to be the set of Beltrami coefficients $\mu$ with support in $\mathbb{H}$ and which are compatible with $G$ in the sense that

$$(5) \quad \mu(g(z))\overline{g'(z)} = \mu(z)g'(z) \quad \text{for all} \quad g \in G.$$  

We assume that $\mu$ is measurable and complex valued and $\|\mu\|_\infty < 1$.

**Theorem 6.1 (Bers).** The Beltrami differentials $\mu$ and $\nu$ of a Riemann surface $S$ are equivalent if and only if $w_\mu = w_\nu$ on $\partial \mathbb{H}$ if and only if $w^\mu = w^\nu$ on $\partial \mathbb{H}$, hence in $L$ if and only if $w_\mu = w_\nu$ in $H$.

**Proof** See Earle [13, p, 152, 153], Gardiner [16], or Lehto [27]. Q.E.D.

**Remark.** For each $\mu \in M(G)$, let $w_\mu : \mathbb{H} \to \mathbb{H}$ be the quasiconformal map that solves Beltrami equation (2) and fixes the three boundary points $\pm 1$ and $i$. Then $w_\mu \circ g \circ (w_\mu)^{-1}$ is a Möbius transformation for each $g$ in $G$. 
Let $G$ be a Fuchsian group and a map $w : \mathbf{H} \to \mathbf{H}$ is a $G$–compatible so that the quasi-conformal deformation $w_{\mu} \circ G \circ w_{\mu}^{-1}$ is also a Fuchsian group.

Two quasiconformal maps $w, \hat{w} : \mathbf{H} \to \mathbf{H}$ are called conformally equivalent if and only if $w|_{\hat{R}} = \hat{w}|_{\hat{R}}$. Then we define the Teichmüller space $T(G)$ of $G$ as

$$T(G) = \{[w]\}.$$

**Theorem 7.1.** If $G$ is finitely generated torsion free, then we may show that

$$T(G) = T(\mathbf{H}/G)$$

for all metrics $d$ [16].

**Remark.** As a consequence of the Theorem 6.1, $T(G)$ is in natural one-to-one correspondence with the set of conformal maps $w^\mu$, $\mu \in M(G)$.

Let $Q$ denote the group of quasiconformal self-mappings of the upper half-plane $H$ and $Q_0$ the (normal) subgroup of $Q$ consisting of elements which keep every $x \in \mathbb{R}$ fixed. And let $N(G)$ be the normalizer of $G$ in $Q$, i.e., the set of $w \in Q$ with $wgw^{-1} = G$.

The *modular group* $\text{mod}(G)$ of $G$ is defined as $\text{mod}(G) = \text{Mod}(G)/G$, where $\text{Mod}(G) = N(G)/((G) \cap Q_0)$.

One verifies that two elements, $[w_1]$ and $[w_2]$, of $T(G)$ are equivalent under $\text{mod}(G)$ if and only if the groups $w_1Gw_1^{-1}$ and $w_2Gw_2^{-1}$ are conjugate in the group of all Möbius transformations. Hence

$$X(G) = T(G)/\text{mod}(G)$$

is the space of conjugacy classes of quasiconformal images of $G$.

If we repeat the preceding definitions, replacing the concept of equivalence ($w_1 \circ w_2^{-1}$ leaves all points of $\hat{R}$ fixed) by that of $G$–equivalence ($w_1 \circ w_2^{-1}$ leaves all points of $\Lambda(G)$ fixed), we arrive at the reduced Teichmüller space $T^\#(G)$ and the reduced modular group $\text{mod}^\#(G)$. As before, there are canonical surjections $\text{mod}(G) \to \text{mod}^\#(G)$, and as before, we have that $X(G) = T^\#(G)/\text{mod}^\#(G)$.

The preceding considerations apply, in particular, to the case $G = 1$, the trivial group. In view of Theorem 7.1, $T(1)$ can be identified with $T(\mathbf{H})$, and $\text{mod}(1)$ with $\text{mod}(\mathbf{H})$. Note also that $T^\#(1)$ reduces to a point.

The definition of Teichmüller space imply at once that for two Fuchsian groups $G_1$ and $G_2$ and $G_1 \subset G_2$, there is an inclusion $T(G_2) \subset T(G_1)$. Hence every $T(G)$ is a subset of the *Universal Teichmüller space* $T(1)$.

**Theorem 7.2.** Every $T(G)$ is closed in $T(1)$ and the inclusion $T(G) \subset T(1)$ is a homeomorphism (Lehto [27]).
Now let us assume that $f$ is holomorphic in a domain $D \subset \mathbb{C}$ and $f'(z) \neq 0$ in $D$. We then define the Schwarzian derivative $S(f)$ of the function $f$ by

$$S(f) = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2.$$

We need the following two properties of Schwarzian derivatives in the sequel. Both are classical and easy to verify.

\begin{enumerate}
  \item \label{eq:6} $S(f \circ g) = (S(f) \circ g)(g')^2 + S(g)$.
  \item \label{eq:7} $S(f) = 0$ if and only if $f$ is a Möbius transformation.
\end{enumerate}

The Schwarzian derivative can be prescribed:

**Theorem 7.3 (Existence and Uniqueness).** Let $\phi$ be a holomorphic function in a simply connected domain $D \subset \mathbb{C}$, then there exists a meromorphic function $f$ in $D$ such that $Sf = \phi$. The solution is unique up to an arbitrary Möbius transformation.

**Proof** See Lehto [27]. \hfill Q.E.D.

Then we may let $T(1)$ be the space of univalent functions which is a quasiconformal extension of lower half-plane, i.e., the set of Schwarzian derivatives of functions admitting an extension to a quasiconformal self-map of $\hat{\mathbb{C}}$. So every element of $T(1)$ can be written as

$$\phi^\mu(z) = S(w^\mu|L), \ \mu(z) : \text{measurable, } \|\mu\|_\infty < 1, \ \mu|_L = 0.$$ 

Therefore $T(1)$ can be embedded in a Banach space consisting of this kind of elements.

The set $T(1)$ turns out to be a domain which is contractible, homogeneous and holomorphically convex. Gehring proved that $T(1)$ is an interior of the closure of $T(1)$.

8. **The Bers Embedding of $T(G)$**

The most interesting case is $T(G)$ to be complex manifold. $T(S) = T(G)$ is a complex manifold if $S$ is a finite quasiconformal type and there is a canonical embedding of $T(G)$ into the Banach space (maybe finite dimensional or maybe not). There are several other ways of giving complex structure on $T(S)$. They are all equivalent. It is at this point that the deeper theory of Teichmüller spaces begins. But we cannot pursue all of these matters here. We here introduce a method by Bers to define local coordinate for $T(S)$.

**Lemma 8.1.** If $\mu$ is in $M(G)$, then $S(w^\mu) = \phi$ is a holomorphic quadratic differential for $G$ in the lower half plane $L$. Moreover $S(w^\mu) = S(w^\nu)$ if and only if $w^\mu = w^\nu$ in $L$.

**Proof** Note that $w^\mu$ is holomorphic in the lower half plane. On taking the Schwarzian derivative of both sides of the equation $w^\mu \circ g = g^\mu \circ w^\mu$ and using the Cayley identity (6), we see that $\phi(gz)g'(z)^2 = \phi(z)$ for $g$ in $G$. To prove the second part, suppose $w^\mu = w^\nu$ in $L$, then
of course $S(w^u) = S(w^v)$. Conversely, suppose $S(w^u) = S(w^v)$. Put $h = w^u \circ (w^v)^{-1}$ in $w^v(L)$. Then in $L$,

$$S(w^u) = S(h \circ w^v) = (S(h) \circ w^v)((w^v)')^2 + S(w^v),$$

so $S(h) = 0$ in $w^v(L)$ and $h$ is a Möbius transformation fixes 3 points. So it is identity. Q.E.D.

Define the Banach space $B(L, G)$ by

$$B(L, G) = \{ \phi(z), \text{ holomorphic } z \in L \text{ with } \| \phi \| = \sup |y^2 \phi(z)| < \infty \},$$

where $\phi(z)dz^2$ is a $G$-invariant, i.e., $\phi(g(z))g'(z)^2 = \phi(z)$. Then the complex dimension of the space $B(L, G)$ is $3g - 3 + n$ if genus $g$ with $n$ points omitted, where $2g - 2 + n > 0$, 1 if $S$ is of type $(1, 0)$, 0 if $S$ is of type $(0, i)$, $i = 0, 1, 2, 3$, and $\infty$ otherwise.

To describe the embedding $T(S) \hookrightarrow B(L, G)$, first extend $\mu$ in $H$ to $C$ by defining $\mu = 0$ in $L$, then $w^u : C \rightarrow C$ is a $\mu$-conformal which fixes 0, 1 and $\infty$. (Note by this that $w^u|_L$ is holomorphic and that it is $w_\mu$.)

Instead to say a Teichmüller space of $w^u|_L$, we can say that $T(G)$ is the set of all Schwarzian derivatives of some locally univalent function $w$ and $|y^2 \phi| \leq \frac{3}{2}$ is satisfied, where $S(w) = \phi$. On the other hand we have the following.

**Lemma 8.2** (Nehari-Kraus). If $f$ is holomorphic and univalent (one-to-one) in the lower half plane, then $|S(f)y^2| \leq \frac{3}{2}$.

**Proof** See Gardiner [16]. Q.E.D.

Also Nehari proved that if $\phi$ satisfies $|y^2 \phi| \leq 1/2$, where $\phi = S(w)$, then $w$ is a schlicht function. At the same time we get

**Lemma 8.3** (Ahlfors-Weill). Suppose $\phi$ is a holomorphic function in the lower half plane $L$ which satisfies $\|\phi(z)y^2\|_\infty \leq \frac{1}{2}$. Let $\mu(z) = -2y^2\phi(z)$ for $z$ in $H$ and $\mu(z) \equiv 0$ for $z \in L$. Then $S(w^u) = \phi$. Moreover, if $\phi$ is a quadratic differential for $G$, then $\mu$ is a Beltrami coefficient in $M(G)$.

**Proof** See Gardiner [16]. Q.E.D.

Combining the previous facts, we have

**Theorem 8.1.** Let $G$ be a Fuchsian group acting on $H$. Then the mapping $\Phi : M(G) \rightarrow B(L, G)$ defined by $\Phi(\mu) = S(w^u)$ induces a one to one mapping $\Phi : T(G) \rightarrow B(L, G)$ whose image in $B(L, G)$ is contained in the ball of radius $\frac{3}{2}$ and contains the ball of radius $\frac{1}{2}$.

**Proof** Gardiner [16]. Q.E.D.
Therefore we have the following theorem which shows that $T(G) = T(S)$ is a complex manifold if $S$ is a quasiconformal type $(g, n)$, $2g - 2 + n > 0$.

**Theorem 8.2.** The Teichmüller space $T(G)$ is a complex manifold modelled on the Banach space $B(L, G)$ if $S$ is a quasiconformal type. When $B(L, G)$ is finite dimensional, then the mapping $\overline{\Phi} : T(G) \to B(L, G)$ is a homeomorphism of $T(G)$ onto a bounded open subset of $B(L, G)$ of complex dimension $3g - 3 + n$ if $2g - 2 + n > 0$. If $S$ is of type $(1, 0)$, then $T(G)$ is holomorphically equivalent to $\Delta$.

**Remark**  
(1) $g$ and $n$ essentially determine the Banach space and Teichmüller space.
(2) The space $B(L, G)$, and hence $T(G)$, is infinite dimensional unless $G$ is finitely generated and of the first kind.
(3) We may identify $T(G)$ with $B_1(L, G)$ which is an open ball of radius 1 in $B(L, G)$.
(4) The reduced Teichmüller space $T^*(G)$ has a canonical real-analytic structure and a canonical embedding as a domain in a real Banach space.

9. Groups on the boundary of $T(G)$

For each $\phi$ in $B(L, G)$ there is a unique locally one-to-one meromorphic function $w_\phi$ in $L$ whose Schwarzian derivative is $\phi$ and whose Laurent expansion in $L$ has the form

$$w_\phi(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}, \quad z \in L.$$  

The function $w_\phi$ depends continuously on $\phi$ in the sense that if a sequence converges in $B(L, G)$ then the corresponding sequence of functions $w_\phi$ converges uniformly on compact sets in $L$, with respect to the spherical metric on $C \cup \{\infty\}$. If $\phi = \Phi(\mu)$ belongs to $T(G)$, then $w_\phi = w^{\mu}$ is a conformal map in $L$.

**Proposition 9.1** (Bers). For each $\phi \in \partial T(G)$, $w_\phi$ is a conformal map of $L$ into $C$, and $G_\phi = w_\phi G(w_\phi)^{-1}$ is a Kleinian group whose regular set contains the simply connected region $w_\phi(L) \cup \{\infty\}$.

**Proof** Choose a sequence $\phi_n = \Phi(\mu_n)$ in $T(G)$ converging to $\phi$. The corresponding conformal maps converge to $w_\phi$, so $w_\phi$ is either conformal or constant. The normalization (8) prevents $w_\phi$ from being constant, so $w_\phi$ is conformal. For each $g$ in $G$ the Möbius transformations $g^{\mu_n}$ converge to $w_\phi \circ g \circ (w_\phi)^{-1}$, so $G_\phi$ is a group of Möbius transformations. It is clear that $w_\phi(L) \cup \{\infty\}$ contains no limit points of $G_\phi$.

**Remark.** If $T(G)$ is finite dimensional, then $w_\phi(L) \cup \{\infty\}$ is a simply connected component of the regular set of $G_\phi$, invariant under $G_\phi$. Finitely generated Kleinian groups whose regular set has a simply connected invariant component are called $B$-groups. They have interesting and important properties. See papers of Bers [9] and Maskit [28] for details.
10. The Bers Fibre Space

For every $\phi = \Phi(\mu)$ in $T(G)$ there exists a well defined Jordan region $D_\phi = w^\mu(L)$ and a well defined conformal map $w_\mu = w^\mu$ of $L$ onto the complement of $D_\phi$. The Quasi-Fuchsian group $G_\phi = w^\mu G(w^\mu)^{-1}$ is also independent of $\mu$ (provided that $\Phi(\mu) = \phi$), since $g_\phi = w_\phi \circ g \circ (w_\phi)^{-1}$ in the complement of $D_\phi$. The quotient space $D_\phi / G_\phi$ is equivalent to the Riemann surface represented by $\phi$ in $T(G)$. With Bers we define

$$F(G) = \{(\phi, z) \in T(G) \times \mathbb{C}; z \in D_\phi\}$$

and we define an action of $G$ on $F(G)$ by

$$g(\phi, z) = (\phi, g_\phi(z)).$$

**Theorem 10.1** (Bers [10]). $F(G)$ is a bounded region in $B(L, G) \times \mathbb{C}$. $G$ is a discontinuous group of biholomorphic maps of $F(G)$ onto itself. The quotient space $V(G) = F(G) / G$ is a complex manifold. The obvious projection $(\phi, z) \mapsto \phi$ of $F(G)$ onto $T(G)$ induces a holomorphic projection $\pi : V(G) \to T(G)$ so that $\pi^{-1}(\phi)$ is the Riemann surface $D_\phi / G_\phi$ for each $\phi$ in $T(G)$.

**Remark.** By definition, a complex analytic family of closed Riemann surfaces consists of a pair of complex manifold $V$ and $B$ and a holomorphic map $\pi$ of $V$ onto $B$ such that:

(i) $\pi$ is a submersion (its differential is surjective everywhere),

(ii) $\pi$ is proper (the inverse image of every compact set is compact).

(iii) $\pi^{-1}(t)$ is a closed Riemann surface for every $t$ in $B$.

If the quotient space $H / G$ is compact, then the above projection $\pi : V(G) \to T(G)$ define a holomorphic family of closed Riemann surfaces of genus $g$ over the Teichmüller space $T(G) = T(S)$. For $n = 0$, Grothendieck ([19]) called attention to a universal property of this family and showed how this property makes possible an axiomatic description and construction of the Teichmüller space of the Riemann surface of type $(g, 0)$.

11. The Teichmüller Theorem

In the early 1940’s, O. Teichmüller proved two theorems which now form the foundation of the deformation theory of Riemann surfaces. They are known as Teichmüller’s Existence and Uniqueness Theorems or, collectively, as Teichmüller’s Theorem. The first proof of the theorem which was acceptable to the mathematics community was given by Ahlfors in 1954 ([5]). Teichmüller’s proof of the uniqueness part is still the most elementary. The easiest existence proof is now based on Bers $\mu-$trick, which we have used in the previous sections to define conformal structures on the Teichmüller space of a surface.

**Definition 11.1.** We call $w_\mu$ a Teichmüller mapping and $\mu$ a Teichmüller differential if and only if $\mu = k\phi / |\phi|$, where $0 \leq k < 1$ and $\phi$ is a holomorphic function in $H$.

**Definition 11.2.** We call the quasiconformal map $w_\mu : H \to H$ extremal if and only if $K(w_\mu) \leq K(w_\nu)$ for all $\nu \in M(G)$ with $w_\mu = w_\nu$ on the real axis.
Theorem 11.1. (Teichmüller’s theorem)

1. Uniqueness: Let \( f_0 : (S_1, f_1) \to (S_2, f_2) \) be a Teichmüller extremal mapping of one marked Riemann surface of type \((g, n)\) onto another such surface. If \( f : (S_1, f_1) \to (S_2, f_2) \) is a mapping distinct from \( f_0 \) but homotopic to \( f_0 \), then

\[ K[f_0] < K[f]. \]

2. Existence: Let \((S_1, f_1)\) and \((S_2, f_2)\) be two marked Riemann surfaces of the same genus \( g > 1 \). Then there is a Teichmüller extremal mapping \( f_0 : (S_1, f_1) \to (S_2, f_2) \).

Proof See Abikoff [1] or Ko [22]. Q.E.D.

In the Riemann surface of type \((1, 0)\), for any marked Riemann surface \((S_1, f_1)\), the map \( f_1 : S \to S_1 \) is homotopic to a map \( f \) covered in \( C = \tilde{S} \) by a real-linear homeomorphism sending the period parallelogram of \( S \) to the period parallelogram of \( S_1 \). We then have the following result.

Theorem 11.2 (Teichmüller map for the torus). In the above notation,

1. If \((S_1, f_1)\) is a marked Riemann surface of compact torus, then \( f_1 \) is the Teichmüller mapping in its homotopy class.

2. Let \((S_1, f_1)\) and \((S_2, f_2)\) be two marked Riemann surfaces of genus 1 with \( f_1 \) and \( f_2 \) Teichmüller maps. Then \( f_0 = f_2 \circ f_1^{-1} \) is the Teichmüller map in the homotopy class of \( f_2 \circ f_1^{-1} \).

Proof See Nag [29, p. 148 - 149]. Q.E.D.

12. The \( \lambda \)-Lemma

In this section we introduce one application of Teichmüller theory which we stated so far. We begin with the following definition.

Let \( E \) be a subset of the Riemann sphere \( \hat{C} = \mathbb{C} \cup \{\infty\} \) containing at least 4 points. Let \( \Delta_r \) denote the open disc \(|z| < r\) in \( \mathbb{C} \). A map

\[ f : \Delta_r \times E \to \hat{C} \]

will be called admissible if \( f(0, z) = z \) for all \( z \in E \), for every fixed \( \lambda \in \Delta_r \) the map \( f(\lambda, \cdot) : E \to \hat{C} \) is an injection, and for every fixed \( z \in E \) the map \( f(\cdot, z) : \Delta_r \to \hat{C} \) is holomorphic (i.e., a meromorphic function of \( \lambda \)).

In other words, an admissible map is a family of injections \( E \to \hat{C} \) holomorphically parametrized by a complex parameter \( \lambda \), \(|\lambda| < r\), which reduces to the identity for \( \lambda = 0 \).

We shall assume that the admissible map considered is normalized, i.e., that \( \{0, 1, \infty\} \subset E \) and \( f(\lambda, \zeta) = \zeta \) for \( \zeta = 0, 1, \infty \) and \( \lambda \in \Delta_r \).
The “\(\lambda\)–lemma” by Mañé, Sad, Sullivan asserts that an admissible map \(f(\lambda, z)\) is, for every fixed \(\lambda\), uniformly continuous in \(z\) (with respect to the spherical metric) and that the continuous extension of \(f(\lambda, \cdot)\) to the closure of \(E\) (in \(\hat{C}\)) is quasiconformal.

**Theorem 12.1** (Bers and Royden). If \(f : \Delta_1 \times E \to \hat{C}\) is admissible, then every \(f(\lambda, \cdot)\) is the restriction to \(E\) of a quasiconformal self-map \(F_\lambda\) of \(\hat{C}\), of dilatation not exceeding,

\[
K = \frac{1 + |\lambda|}{1 - |\lambda|}.
\]

Here \(f(\lambda, z)\) is jointly continuous on \(\Delta_1 \times E\).

**Proof**  
First we prove the theorem for the finite set \(E\). Without loss of generality we assume that

\[
E = \{0, 1, \infty, \zeta_1, \ldots, \zeta_n\}, \quad n > 0,
\]

and that the given admissible map \(f : \Delta_1 \times E \to \hat{C}\) is normalized.

Let \(M_n\) denote the complex manifold of ordered \(n\)–tuples of distinct complex numbers \((z_1, \ldots, z_n)\) none of which equals 0 or 1.

Then there is a holomorphic universal covering

\[
p : T(\hat{C} - E) \to M_n
\]

defined by (proof is omitted here)

\[
[w^\mu] \mapsto p([w^\mu]) = ((w^\mu(\zeta_1)), \ldots, (w^\mu(\zeta_n))).
\]

The given admissible map \(f : \Delta_1 \times E \to \hat{C}\) may be identified with a holomorphic vector-valued map \(\tilde{f} : \Delta_1 \to M_n\) which takes \(\lambda \in \Delta_1\) into

\[
\{f(\lambda, \zeta_1), \ldots, f(\lambda, \zeta_n)\} \in M_n.
\]

This map lifts, via (12), to a holomorphic map

\[
\tilde{f} : \Delta_1 \to T(\hat{C} - E) \subset B(\mathbf{L}, G)
\]

(\(G\) is a torsion-free Fuchsian group with \(\hat{C} - E\) conformal to \(H/G\)). The map \(\tilde{f}\) is uniquely determined by the requirement that \(\tilde{f}(0) = [id]\), i.e., the origin in \(B(\mathbf{L}, G)\).

In \(\Delta_1\) the Kobayashi distance between 0 and \(\lambda\) equals the Poincaré distance \(\log K\), where

\[
K = \frac{1 + |\lambda|}{1 - |\lambda|}.
\]

The holomorphic map \(\tilde{f}\) does not increase the Kobayashi distance so that the Teichmüller (=Kobayashi) distance between the points \([id]\) and \(\tilde{f}(\lambda)\) in \(T(\hat{C} - E)\) is at most \(\log K\). This means that there exists, for each \(\lambda \in \Delta_1\), a \(\nu_\lambda \in L_\infty(\mathbf{C})\), with \(K(w^{\nu_\lambda}) \leq K\), i.e., with \(\|\nu_\lambda\| \leq |\lambda|\) and such that

\[
w^{\nu_\lambda}(\zeta_j) = f(\lambda, \zeta_j), \quad j = 1, \ldots, n.
\]
This completes the case for finite. (Note that we have no reason to assume that \( \nu \lambda \) depends holomorphically on \( \lambda \). Whether it can be so chosen, for all \( |\lambda| < 1 \), is equivalent to the Mañé-Sullivan problem.)

Now let \( f : \Delta_1 \times E \to \hat{C} \) be a normalized admissible map, with \( E \) infinite. Choose a sequence of finite sets \( E_j, j = 1, 2, \ldots \) such that \( \{0, 1, \infty\} \subset E_j \subset E \) for all \( j \) and \( E_1 \cup E_2 \cup \ldots \) is dense in \( E \). For a fixed \( \lambda \in \Delta \), denote by \( F_j\lambda \) a \( K \)-quasiconformal self-map of \( \hat{C} \) such that \( F_j\lambda|E_j = f(\lambda, \cdot)|E_j \), \( K \) being given by (10). Such \( F_j \) exist, since Theorem 12.1 holds for finite \( E \). Since all \( F_j \) fix \( 0, 1, \infty \) and are \( K \)-quasiconformal homeomorphisms, a subsequence converge uniformly (in the spherical metric) to a \( K \)-quasiconformal homeomorphism \( F : \hat{C} \to \hat{C} \) with \( F = f(\lambda, \cdot) \) on \( \cup E_j \).

Had we assumed \( f(\lambda, \cdot) \) to be continuous, we could have concluded that \( F(z) = f(\lambda, z) \) for \( z \in E \), but we made no such assumption. However, let \( c \) be any point in \( E \). Replacing \( E_j \) by \( E_j \cup \{c\} \) and repeating the previous construction we obtain a \( K \)-quasiconformal self-map \( F' \) of \( C \) which coincides with \( f(\lambda, \cdot) \) on \( \cup E_j \cup \{c\} \). But \( F \) and \( F' \) are continuous everywhere and coincide on \( \cup E_j \), hence on \( E \), hence \( F(c) = F'(c) = f(\lambda, \cdot) \). Since \( c \) is arbitrary, \( F|E = f(\lambda, \cdot) \). Theorem is proved. Q.E.D.

It is easy to see that the bound (10) cannot be improved. From Theorem 12.1 we derive the following corollary:

**Corollary 12.1** (Mañé-Sad-Sullivan). If \( f : \Delta_1 \times \hat{C} \to \hat{C} \) is admissible, then for each \( \lambda \in \Delta_1 \), the map \( f(\lambda, \cdot) \) is a quasiconformal homeomorphism of \( \hat{C} \) onto itself.

**Theorem 12.2.** If \( f : \Delta_1 \times E \to \hat{C} \) is admissible and \( E \) has a nonempty interior \( \omega \), then for each \( \lambda \in \Delta_1 \) the map \( f(\lambda, \cdot)|\omega \) is a \( K \)-quasiconformal homeomorphism of \( \omega \) into \( \hat{C} \) with \( K = (1 + |\lambda|)/(1 - |\lambda|) \). The Beltrami coefficient of \( f(\lambda, \cdot)|\omega \) given by

\[
\mu(\lambda, \cdot) = \frac{\partial f(\lambda, z)|\omega}{\partial z} \left/ \frac{\partial f(\lambda, z)|\omega}{\partial z} \right.,
\]

is a holomorphic function of \( \lambda \in \Delta_1 \), qua element of the Banach space \( L_\infty(\omega) \).

Given an admissible map \( f : \Delta_1 \times E \to \hat{C} \) we may want to find an admissible map \( \tilde{f} : \Delta_1 \times \hat{C} \to \hat{C} \) which extends \( f \). This extension problem first posed by Mañé and Sullivan, seems difficult. Bers and Royden [11] obtained the following partial results.

**Proposition 12.1.** If for every finite set \( E_0 \subset \hat{C} \) (containing at least three points) and for every point \( y \notin E_0 \) every admissible map of \( \Delta_1 \times E_0 \) extends to an admissible map of \( \Delta_1 \times (E_0 \cup \{y\}) \), then the extension problem is solvable for any set \( E \) and any admissible map of \( \Delta_1 \times E \).

If \( E \) is a set consisting of three points, then every admissible map \( f \) on \( \Delta_1 \times E \) trivially extends to an admissible map \( \tilde{f} \) on \( \Delta_1 \times \hat{C} \), for we may assume \( f \) normalized and take \( \tilde{f}(\lambda, z) = z \). The corresponding result for a set of four points is given by Proposition 12.2 below which is implied by a result of Earle and Kra. For a set \( E \) with \( n \) points, \( n > 4 \), we do not know whether
every admissible map on \( \Delta_1 \times E \) extends to an admissible map on \( \Delta_1 \times (E \cup \{ y \}) \), for a point \( y \in \mathbb{C} - E \).

**Proposition 12.2.** Let \( E = \{ 0, 1, \infty, \alpha \} \) be a set consisting of four points and \( f : \Delta_1 \times E \rightarrow \hat{\mathbb{C}} \) an admissible map. Then there is an admissible map \( \tilde{f} : \Delta_1 \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \) which extends \( f \).

The “Improved \( \lambda \)–lemma” by Sullivan and Thurston [35] asserts that there is an \( r > 0 \), which they cannot estimate, such that for every admissible map \( f \) on \( \Delta_1 \times E \) there is an admissible map on \( \Delta_r \times \hat{\mathbb{C}} \) which extends \( f |_{\Delta_r \times E} \).

**Theorem 12.3.** If \( f : \Delta_1 \times E \rightarrow \hat{\mathbb{C}} \) is an admissible map, then \( f |_{\Delta_{1/3} \times E} \) has a canonical admissible extension \( \tilde{f} : \Delta_{1/3} \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \).

This extension is characterized by the following property. Let \( \mu(\lambda, z) \) be the Beltrami coefficient of \( z \mapsto \tilde{f}(\lambda, z) \) and \( S \) any component of \( \hat{\mathbb{C}} - \hat{E} \), where \( \hat{E} \) is the closure of \( E \) in \( \hat{\mathbb{C}} \). Then

\[
\mu(\lambda, z) = \rho_S(z)^{-2} \psi(\lambda, z) \quad \text{for} \quad z \in S, \; \lambda \in \Delta_{1/3},
\]

where \( \rho_S(z)|dz| \) is the Poincaré line element in \( S \) and the function \( \psi(\lambda, z) \) is holomorphic in \( z \in S \), antiholomorphic in \( \lambda \in \Delta_{1/3} \).

**Proof** We here prove this theorem for the finite \( E \) only. Observe, in the proof of the Theorem 12.1, that \( \tilde{f} \) maps \( \Delta_1 \) into the ball \( \| \phi \| < \frac{3}{2} \) in the \((n - 3)\)–dimensional Banach space \( B(\mathbf{L}, G) \). By the Schwarz lemma (which is valid for vector-valued functions), \( \tilde{f} \) takes the disc \( |\lambda| < \frac{1}{2} \) into the ball \( \| \phi \| < \frac{1}{2} \). There exists, for each \( \lambda \in \Delta_{1/3} \), a harmonic Beltrami coefficient \( \nu_\lambda \) in \( \hat{\mathbb{C}} - \hat{E} \), which depends holomorphically on \( \tilde{f} \in B(\mathbf{L}, G) \), and hence on \( \lambda \), and such that (14) holds. Since \( w^{\nu_\lambda}(z) \) depends holomorphically on \( \lambda \), the admissible map \( \tilde{f}(\lambda, z) = w^{\nu_\lambda}(z), \; |\lambda| < \frac{1}{2}, \; z \in \hat{\mathbb{C}} \), is the extension of \( f |_{\Delta_{1/3} \times E} \) the existence of which is asserted by the Theorem 12.3. Q.E.D.

The uniqueness statement in Theorem 12.3 is based on a result which may be of interest in other connections, too. It gives a sufficient condition for a quasiconformal self-map of a plane domain which is homotopic to the identity modulo the set-theoretical boundary to be so modulo the ideal boundary.

The proofs of the Theorem 12.2 to Theorem 12.3 make essential use of the theory of quasiconformal mappings and of Teichmüller spaces. Detailed proofs can be found in Bers and Royden [11].

13. Embedding of Riemann surfaces into Riemannian manifold

Finally we examine the embedding problem of Riemann surfaces into Riemannian manifold.
13.1. **The Embedding Problem and its solution.** \(C^\infty\)-embedded surfaces are called *classical surfaces* if they are viewed as Riemann surfaces whose conformal structure is given in the following natural way: the local coordinates are those which preserve angles and orientation.

In 1882, Klein posed the question of whether every Riemann surface is conformally equivalent to a classical surface (see Klein [20, p. x] and [21, p. 635]).

The first non-trivial result was obtained by Teichmüller ([37]). He deformed an embedded surface by moving each point in the normal direction and studied the dependence of the conformal structures of the perturbed surface on deformation parameters.

Around 1960, A. Garsia ([17], [18]) proved that every compact Riemann surface can be conformally immersed in Euclidean 3-space \(\mathbb{R}^3\). He stated that he had found a realization of every compact Riemann surface as a classical surface although Klein required that classical surfaces be embedded. Garsia’s proof uses Teichmüller’s idea, results, and constructions inspired by Nash’s embedding theorem and Brouwer’s fixed point theorem.

In 1970, Rüedy extended Garsia’s result to open Riemann surfaces \(S\) by applying Garsia’s techniques to compact exhaustions of \(S\) ([32]) and later he proved that every compact Riemann surface can be conformally embedded in \(\mathbb{R}^3\) ([33], [34]).

In 1989, author apply Teichmüller theory to prove that we can find a conformally equivalent model surface in an orientable Riemannian manifold \(M\) of dim \(M\geq 3\) for every compact Riemann surface ([22]).

This study was inspired by recent developments in mathematics and particle physics. Embedded Riemann surfaces occur in string theory—the so called *theory of everything*—as the *world sheets*, that is the trajectories, of strings moving in space-time. The strings are permitted to join and separate. In general, these surfaces are non-compact and have positive genus.

Nondegenerate orientable minimal surfaces in Riemannian manifolds also have natural Riemann surface structures. Classical minimal surfaces in 3-manifolds have been used by Meeks and Yau to prove the equivariant version of Dehn’s lemma (see Bass and Morgan [7, p. 153-163]) among other results.

We here introduce the case of a compact Riemann surface \(S\) in a Riemannian manifold.

Let \(M\) be an orientable Riemannian manifold of dim \(M\geq 3\) and let \(S\) be a closed \(C^\infty\)-embedded Riemann surface in \(M\). We briefly examine one method of constructing deformations of \(S\) in \(M\).

Let \(\Gamma : S \leftrightarrow N\mathbb{S} \setminus \Gamma_0\) be a nowhere vanishing smooth section (with unit length) of the normal bundle \(N\mathbb{S}\) of \(S\) in \(M\). Let \(h : S \to (-\epsilon, \epsilon)\) be a \(C^\infty\)-function on \(S\) and call \(\{h(x)\Gamma(x)\}\) a normal vector field on \(S\).

Let \(M_1\) be the subset of \(N\mathbb{S}\) consisting of all pairs \((x, r) := (x, r\Gamma(x))\) for all \(x \in S\), where \(|r| < 2\epsilon\). Then \(M_1\) contains the pair \((x, h(x)\Gamma(x))\). Also let \(M_2\) be the set of all points \(\{y \in M : y = \exp r\Gamma(x), r \in (-2\epsilon, 2\epsilon), (x, r) \in M_1\}\), then \(M_2\) is a Riemannian submanifold of \(M\) for \(\epsilon\) sufficiently small. Again, for sufficiently small \(\epsilon\), the map \(\beta : M_1 \to M_2\), defined by the exponential map \(\beta(x, r) = \exp r\Gamma(x)\), is a diffeomorphism.
By Nash’s Embedding Theorem, there is a \( C^\infty \)-isometric embedding \( j : M \hookrightarrow \mathbb{R}^m \) for some sufficiently large \( m \). This embedding allows us to consider \( S \) and \( M \) as subsets of \( \mathbb{R}^m \).

Assume that \( \tilde{S} \) is the holomorphic universal covering of \( S \). Let \( X : \tilde{S} \to M \subset \mathbb{R}^m \) be a local parametrization of \( S \) in the orientable Riemannian manifold \( M \subset \mathbb{R}^m \).

For \( X(z) \in S \), let

\[
\alpha_{X(z)} : (-2, 2) \to \mathbb{M} \subset \mathbb{R}^m \quad t \mapsto \beta(X(z), th(X(z))) := \exp\left(th(X(z))\Gamma(X(z))\right).
\]

Then \( \alpha_{X(z)}(t) \) is a \( C^\infty \)-curve for which \( \alpha_{X(z)}(0) = X(z) \) and

\[
\alpha_{X(z)}(1) = \beta(X(z), h(X(z))) = \exp h(X(z))\Gamma(X(z)).
\]

Let \( \tilde{\Gamma}(X(z)) \in T_{X(z)}\mathbb{M} \) be a unit tangent vector in \( \mathbb{M} \) to the curve \( \alpha_{X(z)}(t) \) at the point \( X(z) \in S \). We then have

\[
\alpha_{X(z)}(1) = X(z) + h(X(z))\tilde{\Gamma}(X(z)) + O(h^2(X(z))), \quad |h(X(z))| < \epsilon.
\]

Now, for any given sufficiently small \( \epsilon > 0 \), we may define a normal deformation of \( S \).

**Definition 13.1.** In the above notation, a surface \( S_h \hookrightarrow M \) is called an \( \epsilon \)-normal deformation of \( S \) if, for a given small \( \epsilon > 0 \), \( h \) is a \( C^\infty \)-real-valued function on \( S \) such that \( \|h\|_\infty < \epsilon \) and \( S_h \) is defined by the map:

\[
S_h : S \to \mathbb{M} \subset \mathbb{R}^m \quad X(z) \mapsto \alpha_{X(z)}(1) = \beta(X(z), h(X(z)))
\]

\[
= X(z) + h(X(z))\tilde{\Gamma}(X(z)) + O(h^2) \in \mathbb{M}.
\]

The number \( \|h\|_\infty \) is called the size of the deformation \( h \).

We now may state the Embedding Theorem as follows.

**Theorem 13.1** (Embedding Theorem). Assume that \( S \) is a compact Riemann surface \( C^\infty \)-embedded in the orientable Riemannian manifold \( M \) of \( \dim M \geq 3 \). Let \( S_0 \) be any Riemann surface structure on \( S \). If there exists a nowhere vanishing smooth section of the normal bundle \( NS \) of \( S \) in \( M \), then

1. **[Existence of normal deformations of \( S \)]** There exists an \( \epsilon = \epsilon(S) \) so that there is an embedded \( \epsilon \)-normal deformation \( S_h \) of \( S \), of the form given in (15).

2. **[Existence of Conformal Models]** There exists an \( \epsilon \)-normal deformation \( S_h \) of \( S \) which is conformally equivalent to the given Riemann surface \( S_0 \).

The sketch of the proof will be given in the next subsection.

Let \( \mathcal{E} \) be the set of \( \epsilon \)-normal deformations of \( S \) in \( M \subset \mathbb{R}^m \) which are \( C^\infty \)-embedded. We may parametrize the family \( \mathcal{E} \) by setting

\[
\text{Maps}_\epsilon(S) = \{ f : S \to f(S) \subset \mathbb{R}^m \mid \text{defines an } \epsilon-\text{normal deformation of } S \},
\]
then for each $f \in \text{Maps}_\epsilon(S)$, there is exactly one $\epsilon$-normal deformation $f(S) \in \mathcal{E}$ corresponding to this $f$.

We then have the following characterization of maps of $\text{Maps}_\epsilon(S)$ into the Teichmüller space $\mathcal{T}(S)$ of $\mathcal{T}(S)$.

**Theorem 13.2.** The map of the space $\text{Maps}_\epsilon(S)$ into the Teichmüller space $\mathcal{T}(S)$, defined by $f \mapsto [f]$, is continuous in the $C^1$-topology. This map is not continuous in the $C^0$ (uniform)-topology.

**13.2. Sketch of the Proof of the Embedding Theorem.** First of all we need to examine the question of the existence of smooth nowhere vanishing sections of the normal bundle $N_S$ for the Riemann surface $S$ embedded in the Riemannian manifold $M$.

**Lemma 13.1. (Existence of the smooth nowhere vanishing section of $NS$)**

If $M$ is an orientable manifold and $\dim M = 3$ or $\geq 5$, then there always exists a nowhere vanishing smooth section of the normal bundle of $S$ in $M$. If $\dim M = 4$, then there exists a nowhere vanishing smooth section of the normal bundle of $S$ in $M$ if the Euler class of $NS$ vanishes.

The proof of the first part [Existence of normal deformations of $S$] is given by the following lemma.

**Lemma 13.2.** Suppose $S$ is a Riemann surface in $M$. Assume that there exists a smooth nowhere vanishing section $\Gamma$ of the normal bundle $N_S$ of $S$ in $M$. Then there is an $\epsilon_0 > 0$ so that, for all $C^\infty$-functions $h$ defined on $S$ with $\|h\|_\infty < \epsilon_0$, there exists an $\epsilon_0$-normal deformation $S_h$ of $S$ and $S_h$ is a $C^\infty$-embedded surface.

For the proof of the second part, [Existence of conformal models], good estimates of the distance between two points in $\mathcal{T}(S)$, of a compact Riemann surface of $g \geq 1$, will be crucial in our arguments. The following Lemma, due to Garsia ([18]), serves this purpose. In order to formulate it, we have to fix, in the holomorphic universal covering space $\tilde{S}$ of $S$, a fundamental domain $P$ for the covering group $G$. Assume that $\omega \in B(L, G)$ is a local coordinate for a neighborhood of $[\text{id}_S]$ in $\mathcal{T}(S)$ provided $\|\omega\| \leq 2\epsilon < 1$. Let $S_\omega$ be a Riemann surface structure on $S$ with the metric

\[
(15) \quad ds^2_\omega := \lambda^2(z) \left| dz + \frac{\phi_\omega(z)}{\phi_\omega(z)} d\bar{z} \right|^2,
\]

where $\lambda^2$ is a smooth real-valued $(1,1)$-form and $\omega = \phi_\omega(z) dz^2$ is a holomorphic quadratic differential on $S$.

If $f_\omega : S \to S_\omega$ is a quasiconformal map and $[f_\omega] \in \mathcal{T}(S)$, then we write $[f_\omega] = \omega$. Let $f_0 : S \to S_0$ be a homeomorphism so that $[f_0] \in \mathcal{T}(S)$. Assume that $[f_0] = \omega_0$ and denote by $B_\epsilon(\omega_0) \subset B(L, G)$ the set of elements in $\mathcal{T}(S)$ with $\|\omega - \omega_0\| < \epsilon$. 
Lemma 13.3. (Garsia [18]) If \([f_\omega] \in \mathcal{B}_\epsilon(\omega_0)\) and if there is a quasiconformal mapping \(\chi : S_\omega \to S_{\omega'}\), whose dilatation \(K_\chi\) satisfies

1. \(K_\chi \leq K_0\),
2. \(K_\chi \leq 1 + \delta\) except on \(A \subset P\) and
3. \(\text{area } A \leq \eta\),

then there is a constant \(b = b(K_0, \delta, \eta)\) so that

\[\|\omega' - \omega\| \leq b(K_0, \delta, \eta).\]

Further, if \(K_0\) is bounded as \((\delta, \eta) \to (0, 0)\), then \(b(K_0, \delta, \eta) \to 0\).

Proof. See Garsia [18, p. 100 ff]. Q.E.D.

Next we fix a map \(h : S \times \mathcal{B}_\epsilon(\omega_0) \to (-\epsilon, \epsilon)\) so that \(h\) is a \(C^\infty\)-function on \(S\) for each fixed \(\omega\). We then define a map \(\Xi\) of \(\mathcal{B}_\epsilon(\omega_0)\) to \(T(S)\) by

\[\Xi : \mathcal{B}_\epsilon(\omega_0) \to T(S)\]

\[\omega \mapsto [S_\omega].\]

Here the surface \(S_\omega\) is the \(\epsilon\)-normal deformation of \(S\) defined by the map

\[S_\omega := S_{h(\cdot, \omega)} : S \to M \subset \mathbb{R}^m\]

\[X(z) \mapsto \alpha_{X(z)}(1) = \beta(X(z), h(X(z))) = X(z) + h(X(z), \omega)\Gamma(X(z)) + r(h^2),\]

where the remainder term \(r(h^2)\) is \(O(h^2)\). Then, as a consequence of Brouwer’s fixed point theorem, we will have proved the existence of the conformal model if we can prove that, given \([f_\omega] = \omega_0\) and \(\epsilon > 0\), for \(\omega\) in the closed ball \(\mathcal{B}_\epsilon(\omega_0) \subset B(L, G)\), there is a family of deformations \(S_\omega\) of \(S\) depending on parameters \(\omega \in \mathcal{B}_\epsilon(\omega_0)\) so that the following is true.

Lemma 13.4 (Dependence of \(S_\omega\) on Parameters \(\omega\)). In the above notation,

1. \(\Xi : \omega \mapsto [S_\omega]\) is continuous in \(\mathcal{B}_\epsilon(\omega_0)\).
2. \(\|S_\omega - [id_\omega]\| \leq \epsilon, \forall \omega \in \mathcal{B}_\epsilon(\omega_0),\) where \(id_\omega : S \to S_\omega\) is the set-theoretic identity map.

Garsia’s Continuity Lemma (Lemma 13.3) implies that the family \(\{S_\omega\}\) satisfies property (1) if the coefficients of \((dS_\omega)^2\) depend continuously on \((z, \omega) \in \tilde{S} \times \mathcal{B}_\epsilon(\omega_0)\). We may give an explicit formula for the functions \(h(\cdot, \omega)\); from the formulas it follows directly that this property is satisfied.

To prove property (2), we let \(\chi = S_\omega \circ (id_\omega)^{-1} : S_\omega \to S_\omega\), then its dilatation \(K_\chi\) satisfies

\[K^2_\chi = \frac{\sup \frac{(dS_\omega)^2}{ds^2_\omega}}{\inf \frac{(dS_\omega)^2}{ds^2_\omega}},\]

where both the supremum and infimum are taken over all directions and \(ds^2_\omega\) is as defined in (15). We can construct the set \(A\) and determine the constants \(\delta, \eta\) and the constant \(\epsilon\), for
which we get $b(K_0, \delta, \eta) \leq \epsilon$ in Garsia’s Continuity Lemma 13.3. Then application of Garsia’s Continuity Lemma gives property (2).

Continuation of the outline of the proof of the existence of the conformal model

By Lemma 13.4, the function $\Xi$ satisfies the hypotheses of the Brouwer’s fixed point theorem. Therefore there is a point $\omega_1 \in B_\epsilon(\omega_0)$ so that

$$\Xi(\omega_1) = [S^{\omega_1}] = \omega_0 = [f_0],$$

where $f_0 : S \to S_0$, i.e., for this $\omega_1 \in B_\epsilon(\omega_0)$, the deformed surface $S^{\omega_1}$ can be mapped conformally onto $S_0$ by a mapping homotopic to $f_0 \circ (S^{\omega_1})^{-1}$.

Final Remarks. Recently this author try to extend this embedding for a Riemann surface of topological type. The answer is positive. This embedding problem may be transfered into the Semi-Riemannian manifold. The ultimate goal is to find a model space for the string theory and to show that Nambu-Goto and Polyakov action in the string theory is the same independent of the ambient manifold.

References


Department of Applied Mathematics
Kon-Kuk University
322 Danwoldong Chungjusi Chungbuk
Seoul Korea 380-701